Microwave

Devices

and Circuits

Third Edition

SAMUEL Y. LIAO

Professor of Electrical Engineering
California State University, Fresno

Chapter 4

Microwave Waveguides and Components

4-0 INTRODUCTION

In general, a waveguide consists of a hollow metallic tube of a rectangular or circular shape used to guide an electromagnetic wave. Waveguides are used principally at frequencies in the microwave range; inconveniently large guides would be required to transmit radio-frequency power at longer wavelengths. At frequency range X band from 8.00 to 12.0 GHz, for example, the U.S. standard rectangular waveguide WR-90 has an inner width of 2.286 cm (0.9 in.) and an inner height of 1.016 cm (0.4 in.); but its outside dimensions are 2.54 cm (1 in.) wide and 1.27 cm (0.5 in.) high [1].

In waveguides the electric and magnetic fields are confined to the space within the guides. Thus no power is lost through radiation, and even the dielectric loss is negligible, since the guides are normally air-filled. However, there is some power loss as heat in the walls of the guides, but the loss is very small.

It is possible to propagate several modes of electromagnetic waves within a waveguide. These modes correspond to solutions of Maxwell's equations for particular waveguides. A given waveguide has a definite cutoff frequency for each allowed mode. If the frequency of the impressed signal is above the cutoff frequency for a given mode, the electromagnetic energy can be transmitted through the guide for that particular mode without attenuation. Otherwise the electromagnetic energy with a frequency below the cutoff frequency for that particular mode will be attenuated to a negligible value in a relatively short distance. *The dominant mode in a particular guide is the mode having the lowest cutoff frequency*. It is advisable to choose the dimensions of a guide in such a way that, for a given input signal, only the energy of the dominant mode can be transmitted through the guide.

The process of solving the waveguide problems may involve three steps:

1. The desired wave equations are written in the form of either rectangular or cylindrical coordinate systems suitable to the problem at hand.
2. The boundary conditions are then applied to the wave equations set up in step 1.
3. The resultant equations usually are in the form of partial differential equations in either time or frequency domain. They can be solved by using the proper method.

4-1 RECTANGULAR WAVEGUIDES

A rectangular waveguide is a hollow metallic tube with a rectangular cross section. The conducting walls of the guide confine the electromagnetic fields and thereby guide the electromagnetic wave. A number of distinct field configurations or modes can exist in waveguides. When the waves travel longitudinally down the guide, the plane waves are reflected from wall to wall. This process results in a component of either electric or magnetic field in the direction of propagation of the resultant wave; therefore the wave is no longer a *transverse electromagnetic* (TEM) wave. Figure 4-1-1 shows that any uniform plane wave in a lossless guide may be resolved into TE and TM waves.
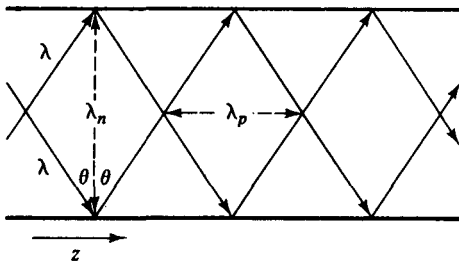

Figure 4-1-1 Plane wave reflected in a waveguide.

It is clear that when the wavelength λ is in the direction of propagation of the incident wave, there will be one component λ_n in the direction normal to the reflecting plane and another λ_p parallel to the plane. These components are

$$\lambda_n = \frac{\lambda}{\cos \theta} \qquad (4\text{-}1\text{-}1)$$

$$\lambda_p = \frac{\lambda}{\sin \theta} \qquad (4\text{-}1\text{-}2)$$

where θ = angle of incidence
λ = wavelength of the impressed signal in unbounded medium

A plane wave in a waveguide resolves into two components: one standing wave in the direction normal to the reflecting walls of the guide and one traveling wave in the direction parallel to the reflecting walls. In lossless waveguides the modes may be classified as either *transverse electric* (TE) mode or *transverse magnetic* (TM) mode. In rectangular guides the modes are designated TE_{mn} or TM_{mn}. The integer m

denotes the number of half waves of electric or magnetic intensity in the x direction, and n is the number of half waves in the y direction if the propagation of the wave is assumed in the positive z direction.

4-1-1 Solutions of Wave Equations in Rectangular Coordinates

As stated previously, there are time-domain and frequency-domain solutions for each wave equation. However, for the simplicity of the solution to the wave equation in three dimensions plus a time-varying variable, only the sinusoidal steady-state or the frequency-domain solution will be given. A rectangular coordinate system is shown in Fig. 4-1-2.
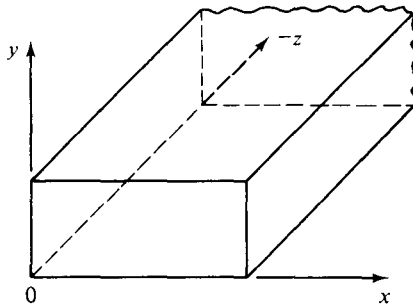

Figure 4-1-2   Rectangular coordinates.

The electric and magnetic wave equations in frequency domain in Eqs. (2-1-20) and (2-1-21) are given by

$$\nabla^2 \mathbf{E} = \gamma^2 \mathbf{E} \tag{4-1-3}$$

$$\nabla^2 \mathbf{H} = \gamma^2 \mathbf{H} \tag{4-1-4}$$

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$. These are called the *vector wave equations*.

Rectangular coordinates are the usual right-hand system. The rectangular components of $\mathbf{E}$ or $\mathbf{H}$ satisfy the complex scalar wave equation or Helmholtz equation

$$\nabla^2 \psi = \gamma^2 \psi \tag{4-1-5}$$

The Helmholtz equation in rectangular coordinates is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \gamma^2 \psi \tag{4-1-6}$$

This is a linear and inhomogeneous partial differential equation in three dimensions. By the method of separation of variables, the solution is assumed in the form of

$$\psi = X(x)Y(y)Z(z) \tag{4-1-7}$$

where $X(x)$ = a function of the x coordinate only
$\quad\quad\; Y(y)$ = a function of the y coordinate only
$\quad\quad\; Z(z)$ = a function of the z coordinate only

Substitution of Eq. (4-1-7) in Eq. (4-1-6) and division of the resultant by Eq. (4-1-7) yield

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \qquad (4\text{-}1\text{-}8)$$

Since the sum of the three terms on the left-hand side is a constant and each term is independently variable, it follows that each term must be equal to a constant.

Let the three terms be k_x^2, k_y^2, and k_z^2, respectively; then the separation equation is given by

$$-k_x^2 - k_y^2 - k_z^2 = \gamma^2 \qquad (4\text{-}1\text{-}9)$$

The general solution of each differential equation in Eq. (4-1-8)

$$\frac{d^2X}{dx^2} = -k_x^2X \qquad (4\text{-}1\text{-}10)$$

$$\frac{d^2Y}{dy^2} = -k_y^2Y \qquad (4\text{-}1\text{-}11)$$

$$\frac{d^2Z}{dz^2} = -k_z^2Z \qquad (4\text{-}1\text{-}12)$$

will be in the form of

$$X = A \sin (k_x x) + B \cos (k_x x) \qquad (4\text{-}1\text{-}13)$$

$$Y = C \sin (k_y y) + D \cos (k_y y) \qquad (4\text{-}1\text{-}14)$$

$$Z = E \sin (k_z z) + F \cos (k_z z) \qquad (4\text{-}1\text{-}15)$$

The total solution of the Helmholtz equation in rectangular coordinates is

$$\psi = [A \sin (k_x x) + B \cos (k_x x)][C \sin (k_y y) + D \cos (k_y y)]$$
$$\times [E \sin (k_z z) + F \cos (k_z z)] \qquad (4\text{-}1\text{-}16)$$

The propagation of the wave in the guide is conventionally assumed in the positive z direction. It should be noted that the propagation constant γ_g in the guide differs from the intrinsic propagation constant γ of the dielectric. Let

$$\gamma_g^2 = \gamma^2 + k_x^2 + k_y^2 = \gamma^2 + k_c^2 \qquad (4\text{-}1\text{-}17)$$

where $k_c = \sqrt{k_x^2 + k_y^2}$ is usually called the *cutoff wave number*. For a lossless dielectric, $\gamma^2 = -\omega^2\mu\epsilon$. Then

$$\gamma_g = \pm\sqrt{\omega^2\mu\epsilon - k_c^2} \qquad (4\text{-}1\text{-}18)$$

There are three cases for the propagation constant γ_g in the waveguide.

Case I.   There will be no wave propagation (evanescence) in the guide if $\omega_c^2\mu\epsilon = k_c^2$ and $\gamma_g = 0$. This is the critical condition for cutoff propagation. The cutoff frequency is expressed as

$$f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{k_x^2 + k_y^2} \qquad (4\text{-}1\text{-}19)$$

Case II.   The wave will be propagating in the guide if $\omega^2\mu\epsilon > k_c^2$ and

$$\gamma_g = \pm j\beta_g = \pm j\omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (4\text{-}1\text{-}20)$$

This means that the operating frequency must be above the cutoff frequency in order for a wave to propagate in the guide.

Case III.   The wave will be attenuated if $\omega^2\mu\epsilon < k_c^2$ and

$$\gamma_g = \pm\alpha_g = \pm\omega\sqrt{\mu\epsilon}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} \qquad (4\text{-}1\text{-}21)$$

This means that if the operating frequency is below the cutoff frequency, the wave will decay exponentially with respect to a factor of $-\alpha_g z$ and there will be no wave propagation because the propagation constant is a real quantity. Therefore the solution to the Helmholtz equation in rectangular coordinates is given by

$$\psi = [A \sin(k_x x) + B \cos(k_x x)][C \sin(k_y y) + D \cos(k_y y)]e^{-j\beta_g z} \qquad (4\text{-}1\text{-}22)$$

4-1-2 TE Modes in Rectangular Waveguides

It has been previously assumed that the waves are propagating in the positive z direction in the waveguide. Figure 4-1-3 shows the coordinates of a rectangular waveguide.
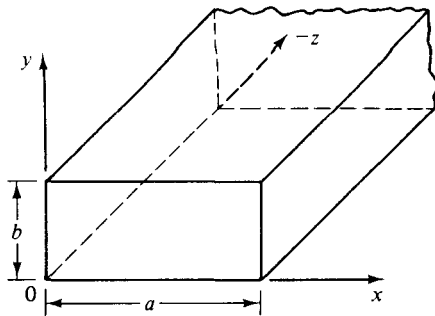

Figure 4-1-3   Coordinates of a rectangular guide.

The TE_{mn} modes in a rectangular guide are characterized by $E_z = 0$. In other words, the z component of the magnetic field, H_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation,

$$\nabla^2 H_z = \gamma^2 H_z \qquad (4\text{-}1\text{-}23)$$

a solution in the form of

$$H_z = \left[A_m \sin\left(\frac{m\pi x}{a}\right) + B_m \cos\left(\frac{m\pi x}{a}\right) \right] \times \left[C_n \sin\left(\frac{n\pi y}{b}\right) + D_n \cos\left(\frac{n\pi y}{b}\right) \right] e^{-j\beta_g z}$$

$$(4\text{-}1\text{-}24)$$

will be determined in accordance with the given boundary conditions, where $k_x = m\pi/a$ and $k_y = n\pi/b$ are replaced. For a lossless dielectric, Maxwell's curl equations in frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mu\,\mathbf{H} \tag{4-1-25}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\,\mathbf{E} \tag{4-1-26}$$

In rectangular coordinates, their components are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \tag{4-1-27}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{4-1-28}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{4-1-29}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \tag{4-1-30}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \tag{4-1-31}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \tag{4-1-32}$$

With the substitution $\partial/\partial z = -j\beta_g$ and $E_z = 0$, the foregoing equations are simplified to

$$\beta_g E_y = -\omega\mu H_x \tag{4-1-33}$$

$$\beta_g E_x = \omega\mu H_y \tag{4-1-34}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} = -j\omega\mu H_z \tag{4-1-35}$$

$$\frac{\partial H_z}{\partial y} + j\beta_g H_y = j\omega\epsilon E_x \tag{4-1-36}$$

$$-j\beta_g H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \tag{4-1-37}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \tag{4-1-38}$$

Solving these six equations for E_x, E_y, H_x, and H_y in terms of H_z will give the TE-

mode field equations in rectangular waveguides as

$$E_x = \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} \tag{4-1-39}$$

$$E_y = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} \tag{4-1-40}$$

$$E_z = 0 \tag{4-1-41}$$

$$H_x = \frac{-j\beta_g}{k_c^2} \frac{\partial H_z}{\partial x} \tag{4-1-42}$$

$$H_y = \frac{-j\beta_g}{k_c^2} \frac{\partial H_z}{\partial y} \tag{4-1-43}$$

$$H_z = \text{Eq. (4-1-24)} \tag{4-1-44}$$

where $k_c^2 = \omega^2\mu\epsilon - \beta_g^2$ has been replaced.

Differentiating Eq. (4-1-24) with respect to x and y and then substituting the results in Eqs. (4-1-39) through (4-1-43) yield a set of field equations. The boundary conditions are applied to the newly found field equations in such a manner that either the tangent **E** field or the normal **H** field vanishes at the surface of the conductor. Since $E_x = 0$, then $\partial H_z/\partial y = 0$ at $y = 0, b$. Hence $C_n = 0$. Since $E_y = 0$, then $\partial H_z/\partial x = 0$ at $x = 0, a$. Hence $A_m = 0$.

It is generally concluded that the normal derivative of H_z must vanish at the conducting surfaces—that is,

$$\frac{\partial H_z}{\partial n} = 0 \tag{4-1-45}$$

at the guide walls. Therefore the magnetic field in the positive z direction is given by

$$H_z = H_{0z} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-46}$$

where H_{0z} is the amplitude constant.

Substitution of Eq. (4-1-46) in Eqs. (4-1-39) through (4-1-43) yields the TE$_{mn}$ field equations in rectangular waveguides as

$$E_x = E_{0x} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-47}$$

$$E_y = E_{0y} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-48}$$

$$E_z = 0 \tag{4-1-49}$$

$$H_x = H_{0x} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-50}$$

$$H_y = H_{0y} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-51}$$

$$H_z = \text{Eq. (4-1-46)} \tag{4-1-52}$$

where $m = 0, 1, 2, \ldots$
$\quad\quad\; n = 0, 1, 2, \ldots$
$\quad\quad\; m = n = 0$ excepted

The cutoff wave number k_c, as defined by Eq. (4-1-17) for the TE$_{mn}$ modes, is given by

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \omega_c \sqrt{\mu\epsilon} \tag{4-1-53}$$

where a and b are in meters. The cutoff frequency, as defined in Eq. (4-1-19) for the TE$_{mn}$ modes, is

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{4-1-54}$$

The propagation constant (or the phase constant here) β_g, as defined in Eq. (4-1-18), is expressed by

$$\beta_g = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-1-55}$$

The phase velocity in the positive z direction for the TE$_{mn}$ modes is shown as

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-56}$$

where $v_p = 1/\sqrt{\mu\epsilon}$ is the phase velocity in an unbounded dielectric.

The characteristic wave impedance of TE$_{mn}$ modes in the guide can be derived from Eqs. (4-1-33) and (4-1-34):

$$Z_g = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-57}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric. The wavelength λ_g in the guide for the TE$_{mn}$ modes is given by

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-58}$$

where $\lambda = v_p/f$ is the wavelength in an unbounded dielectric.

Since the cutoff frequency shown in Eq. (4-1-54) is a function of the modes and guide dimensions, the physical size of the waveguide will determine the propagation of the modes. Table 4-1-1 tabulates the ratio of cutoff frequency of some modes with respect to that of the dominant mode in terms of the physical dimension.

Whenever two or more modes have the same cutoff frequency, they are said to be *degenerate modes*. In a rectangular guide the corresponding TE$_{mn}$ and TM$_{mn}$ modes are always degenerate. In a square guide the TE$_{mn}$, TE$_{nm}$, TM$_{mn}$, and TM$_{nm}$ modes form a foursome of degeneracy. Rectangular guides ordinarily have dimensions of $a = 2b$ ratio. The mode with the lowest cutoff frequency in a particular

TABLE 4-1-1   MODES OF $(f_c)_{mn}/f_c$ FOR $a \geq b$

Modes f/f_{10} a/b	TE_{10}	TE_{01}	TE_{11} TM_{11}	TE_{20}	TE_{02}	TE_{21} TM_{21}	TE_{12} TM_{12}	TE_{22} TM_{22}	TE_{30}
1	1	1	1.414	2	2	2.236	2.236	2.828	3
1.5	1	1.5	1.803	2	3	2.500	3.162	3.606	3
2	1	2	2.236	2	4	2.828	4.123	4.472	3
3	1	3	3.162	2	6	3.606	6.083	6.325	3
∞	1	∞	∞	2	∞	∞	∞	∞	3

guide is called the *dominant mode*. The dominant mode in a rectangular guide with $a > b$ is the TE_{10} mode. Each mode has a specific mode pattern (or field pattern).

It is normal for all modes to exist simultaneously in a given waveguide. The situation is not very serious, however. Actually, only the dominant mode propagates, and the higher modes near the sources or discontinuities decay very fast.

Example 4-1-1: TE_{10} in Rectangular Waveguide

An air-filled rectangular waveguide of inside dimensions 7×3.5 cm operates in the dominant TE_{10} mode as shown in Fig. 4-1-4.
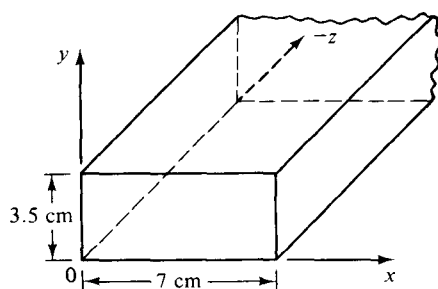

Figure 4-1-4   Rectangular waveguide for Example 4-1-1.

a. Find the cutoff frequency.

b. Determine the phase velocity of the wave in the guide at a frequency of 3.5 GHz.

c. Determine the guided wavelength at the same frequency.

Solution

a. $f_c = \dfrac{c}{2a} = \dfrac{3 \times 10^8}{2 \times 7 \times 10^{-2}} = 2.14$ GHz

b. $v_g = \dfrac{c}{\sqrt{1 - (f_c/f)^2}} = \dfrac{3 \times 10^8}{\sqrt{1 - (2.14/3.5)^2}} = 3.78 \times 10^8$ m/s

c. $\lambda_g = \dfrac{\lambda_0}{\sqrt{1 - (f_c/f)^2}} = \dfrac{3 \times 10^8/(3.5 \times 10^9)}{\sqrt{1 - (2.14/3.5)^2}} = 10.8$ cm

4-1-3 TM Modes in Rectangular Waveguides

The TM_{mn} modes in a rectangular guide are characterized by $H_z = 0$. In other words, the z component of an electric field E must exist in order to have energy transmission in the guide. Consequently, the Helmholtz equation for E in the rectangular coordinates is given by

$$\nabla^2 E_z = \gamma^2 E_z \tag{4-1-59}$$

A solution of the Helmholtz equation is in the form of

$$E_z = \left[A_m \sin \left(\frac{m\pi x}{a} \right) + B_m \cos \left(\frac{m\pi x}{a} \right) \right] \left[C_n \sin \left(\frac{n\pi y}{b} \right) + D_n \cos \left(\frac{n\pi y}{b} \right) \right] e^{-j\beta_g z}$$
$$\tag{4-1-60}$$

which must be determined according to the given boundary conditions. The procedures for doing so are similar to those used in finding the TE-mode wave.

The boundary conditions on E_z require that the field vanishes at the waveguide walls, since the tangent component of the electric field E_z is zero on the conducting surface. This requirement is that for $E_z = 0$ at $x = 0$, a, then $B_m = 0$, and for $E_z = 0$ at $y = 0$, b, then $D_n = 0$. Thus the solution as shown in Eq. (4-1-60) reduces to

$$E_z = E_{0z} \sin \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right) e^{-j\beta_g z} \tag{4-1-61}$$

where $m = 1, 2, 3, \ldots$
        $n = 1, 2, 3, \ldots$

If either $m = 0$ or $n = 0$, the field intensities all vanish. So there is no TM_{01} or TM_{10} mode in a rectangular waveguide, which means that TE_{10} is the dominant mode in a rectangular waveguide for $a > b$. For $H_z = 0$, the field equations, after expanding $\nabla \times \mathbf{H} = j\omega\epsilon\,\mathbf{E}$, are given by

$$\frac{\partial E_z}{\partial y} + j\beta_g E_y = -j\omega\mu H_x \tag{4-1-62}$$

$$j\beta_g E_x + \frac{\partial E_z}{\partial x} = j\omega\mu H_y \tag{4-1-63}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{4-1-64}$$

$$\beta_g H_y = \omega\epsilon E_x \tag{4-1-65}$$

$$-\beta_g H_x = \omega\epsilon E_y \tag{4-1-66}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \tag{4-1-67}$$

These equations can be solved simultaneously for E_x, E_y, H_x, and H_y in terms of E_z.

The resultant field equations for TM modes are

$$E_x = \frac{-j\beta_g}{k_c^2} \frac{\partial E_z}{\partial x} \tag{4-1-68}$$

$$E_y = \frac{-j\beta_g}{k_c^2} \frac{\partial E_z}{\partial y} \tag{4-1-69}$$

$$E_z = \text{Eq. (4-1-61)} \tag{4-1-70}$$

$$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y} \tag{4-1-71}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} \tag{4-1-72}$$

$$H_z = 0 \tag{4-1-73}$$

where $\beta_g^2 - \omega^2\mu\epsilon = -k_c^2$ is replaced.

Differentiating Eq. (4-1-61) with respect to x or y and substituting the results in Eqs. (4-1-68) through (4-1-73) yield a new set of field equations. The TM$_{mn}$ mode field equations in rectangular waveguides are

$$E_x = E_{0x} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-74}$$

$$E_y = E_{0y} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-75}$$

$$E_z = \text{Eq. (4-1-61)} \tag{4-1-76}$$

$$H_x = H_{0x} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-77}$$

$$H_y = H_{0y} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-78}$$

$$H_z = 0 \tag{4-1-79}$$

Some of the TM-mode characteristic equations are identical to those of the TE modes, but some are different. For convenience, all are shown here:

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{4-1-80}$$

$$\beta_g = \omega\sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-1-81}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-82}$$

$$v_g = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-83}$$

$$Z_g = \frac{\beta_g}{\omega\epsilon} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-1-84}$$

4-1-4 Power Transmission in Rectangular Waveguides

The power transmitted through a waveguide and the power loss in the guide walls can be calculated by means of the complex Poynting theorem described in Chapter 2. It is assumed that the guide is terminated in such a way that there is no reflection from the receiving end or that the guide is infinitely long compared with the wavelength. From the Poynting theorem in Section 2-2, the power transmitted through a guide is given by

$$P_{tr} = \oint \mathbf{p} \cdot d\mathbf{s} = \oint \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \tag{4-1-85}$$

For a lossless dielectric, the time-average power flow through a rectangular guide is given by

$$P_{tr} = \frac{1}{2Z_g}\int_a |E|^2 \, da = \frac{Z_g}{2}\int_a |H|^2 \, da \tag{4-1-86}$$

where $Z_g = \dfrac{E_x}{H_y} = -\dfrac{E_y}{H_x}$
$|E|^2 = |E_x|^2 + |E_y|^2$
$|H|^2 = |H_x|^2 + |H_y|^2$

For TE_{mn} modes, the average power transmitted through a rectangular waveguide is given by

$$P_{tr} = \frac{\sqrt{1 - (f_c/f)^2}}{2\eta}\int_0^b\int_0^a (|E_x|^2 + |E_y|^2) \, dx \, dy \tag{4-1-87}$$

For TM_{mn} modes, the average power transmitted through a rectangular waveguide is given by

$$P_{tr} = \frac{1}{2\eta\sqrt{1 - (f_c/f)^2}}\int_0^b\int_0^a (|E_x|^2 + |E_y|^2) \, dx \, dy \tag{4-1-88}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

4-1-5 Power Losses in Rectangular Waveguides

There are two types of power losses in a rectangular waveguide:

1. Losses in the dielectric
2. Losses in the guide walls

First we shall consider power losses caused by dielectric attenuation. In a low-

loss dielectric (that is, $\sigma \ll \mu\epsilon$), the propagation constant for a plane wave traveling in an unbounded lossy dielectric is given in Eq. (2-5-20) by

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{\eta\sigma}{2} \tag{4-1-89}$$

The attenuation caused by the low-loss dielectric in the rectangular waveguide for the TE_{mn} or TM_{mn} modes is given by

$$\alpha_g = \frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} \qquad \text{for TE mode} \tag{4-1-90}$$

$$\alpha_g = \frac{\sigma\eta}{2}\sqrt{1 - (f_c/f)^2} \qquad \text{for TM mode} \tag{4-1-90a}$$

As $f \gg f_c$, the attenuation constant in the guide approaches that for the unbounded dielectric given by Eq. (4-1-89). However, if the operating frequency is way below the cutoff frequency, $f \ll f_c$, the attenuation constant becomes very large and non-propagation occurs.

     Now we shall consider power losses caused by the guide walls. When the electric and magnetic intensities propagate through a lossy waveguide, their magnitudes may be written

$$|E| = |E_{0z}|e^{-\alpha_g z} \tag{4-1-91}$$

$$|H| = |H_{0z}|e^{-\alpha_g z} \tag{4-1-92}$$

where E_{0z} and H_{0z} are the field intensities at $z = 0$. It is interesting to note that, for a low-loss guide, the time-average power flow decreases proportionally to $e^{-2\alpha_g z}$. Hence

$$P_{tr} = (P_{tr} + P_{loss})e^{-2\alpha_g z} \tag{4-1-93}$$

For $P_{loss} \ll P_{tr}$ and $2\alpha_g z \ll 1$,

$$\frac{P_{loss}}{P_{tr}} + 1 = 1 + 2\alpha_g z \tag{4-1-94}$$

Finally,

$$\alpha_g = \frac{P_L}{2P_{tr}} \tag{4-1-95}$$

where P_L is the power loss per unit length. Consequently, the attenuation constant of the guide walls is equal to the ratio of the power loss per unit length to twice the power transmitted through the guide.

     Since the electric and magnetic field intensities established at the surface of a low-loss guide wall decay exponentially with respect to the skin depth while the waves progress into the walls, it is better to define a surface resistance of the guide walls as

$$R_s \equiv \frac{\rho}{\delta} = \frac{1}{\sigma\delta} = \frac{\alpha_g}{\sigma} = \sqrt{\frac{\pi f \mu}{\sigma}} \qquad \Omega/\text{square} \tag{4-1-96}$$

where ρ = resistivity of the conducting wall in ohms-meter
   σ = conductivity in mhos per meter
   δ = skin depth or depth of penetration in meters

   The power loss per unit length of guide is obtained by integrating the power density over the surface of the conductor corresponding to the unit length of the guide. This is

$$P_L = \frac{R_s}{2} \int_s |H_t|^2 \, ds \qquad \text{W/unit length} \qquad (4\text{-}1\text{-}97)$$

where H_t is the tangential component of magnetic intensity at the guide walls.
   Substitution of Eqs. (4-1-86) and (4-1-97) in Eq. (4-1-95) yields

$$\alpha_g = \frac{R_s \int_s |H_t|^2 \, ds}{2Z_g \int_a |H|^2 \, da} \qquad (4\text{-}1\text{-}98)$$

where

$$|H|^2 = |H_z|^2 + |H_y|^2 \qquad (4\text{-}1\text{-}99)$$

$$|H_t|^2 = |H_{tx}|^2 + |H_{ty}|^2 \qquad (4\text{-}1\text{-}100)$$

Example 4-1-2:   TE$_{10}$ Mode in Rectangular Waveguide

An airfilled waveguide with a cross section 2×1 cm transports energy in the TE$_{10}$ mode at the rate of 0.5 hp. The impressed frequency is 30 GHz. What is the peak value of the electric field occurring in the guide? (Refer to Fig. 4-1-5.)

Solution   The field components of the dominant mode TE$_{10}$ can be obtained by substituting $m = 1$ and $n = 0$ in Eqs. (4-1-47) through (4-1-52). Then

$$E_x = 0 \qquad\qquad H_x = \frac{E_{0y}}{Z_g} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_g z}$$

$$E_y = E_{0y} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_g z} \qquad H_y = 0$$

$$E_z = 0 \qquad\qquad H_z = H_{0z} \cos\left(\frac{\pi x}{a}\right) e^{-j\beta_g z}$$
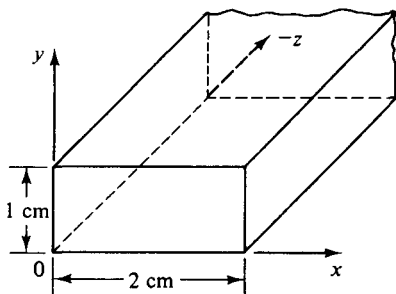
where $Z_g = \omega \mu_0 / \beta_g$.

Figure 4-1-5   Rectangular waveguide for Example 4-1-2.

The phase constant β_g can be found from

$$\beta_g = \sqrt{\omega^2 \mu_0 \epsilon_0 - \frac{\pi^2}{a^2}} = \pi \sqrt{\frac{(2f)^2}{c^2} - \frac{1}{a^2}} = \pi \sqrt{\frac{4 \times 9 \times 10^{20}}{9 \times 10^{16}} - \frac{1}{4 \times 10^{-4}}}$$

$$= 193.5\pi = 608.81 \text{ rad/m}$$

The power delivered in the z direction by the guide is

$$P = \text{Re} \left[\frac{1}{2} \int_0^b \int_0^a (\mathbf{E} \times \mathbf{H}^*) \right] \cdot dx \, dy \, \mathbf{u}_z$$

$$= \frac{1}{2} \int_0^b \int_0^a \left[\left(E_{0y} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_g z} \mathbf{u}_y \right) \times \left(\frac{-\beta_g}{\omega \mu_0} E_{0y} \sin\left(\frac{\pi x}{a}\right) e^{+j\beta_g z} \mathbf{u}_x \right) \right] \cdot dx \, dy \, \mathbf{u}_z$$

$$= \frac{1}{2} E_{0y}^2 \frac{\beta_g}{\omega \mu_0} \int_0^b \int_0^a \left(\sin\left(\frac{\pi x}{a}\right) \right)^2 dx \, dy$$

$$= \frac{1}{4} E_{0y}^2 \frac{\beta_g}{\omega \mu_0} ab$$

$$373 = \frac{1}{4} E_{0y}^2 \frac{193.5\pi (10^{-2})(2 \times 10^{-2})}{2\pi (3 \times 10^{10})(4\pi \times 10^7)}$$

$$E_{0y} = 53.87 \text{ kV/m}$$

The peak value of the electric intensity is 53.87 kV/m.

4-1-6 Excitations of Modes in Rectangular Waveguides

In general, the field intensities of the desired mode in a waveguide can be established by means of a probe or loop-coupling device. The probe may be called a monopole antenna; the coupling loop, the loop antenna. A probe should be located so as to excite the electric field intensity of the mode, and a coupling loop in such a way as to generate the magnetic field intensity for the desired mode. If two or more probes or loops are to be used, care must be taken to ensure the proper phase relationship between the currents in the various antennas. This factor can be achieved by inserting additional lengths of transmission line in one or more of the antenna feeders. Impedance matching can be accomplished by varying the position and depth of the antenna in the guide or by using impedance-matching stubs on the coaxial line feeding the waveguide. A device that excites a given mode in the guide can also serve reciprocally as a receiver or collector of energy for that mode. The methods of excitation for various modes in rectangular waveguides are shown in Fig. 4-1-6.

In order to excite a TE_{10} mode in one direction of the guide, the two exciting antennas should be arranged in such a way that the field intensities cancel each other in one direction and reinforce in the other. Figure 4-1-7 shows an arrangement for launching a TE_{10} mode in one direction only. The two antennas are placed a quarter-wavelength apart and their phases are in time quadrature. Phasing is compensated by use of an additional quarter-wavelength section of line connected to the antenna
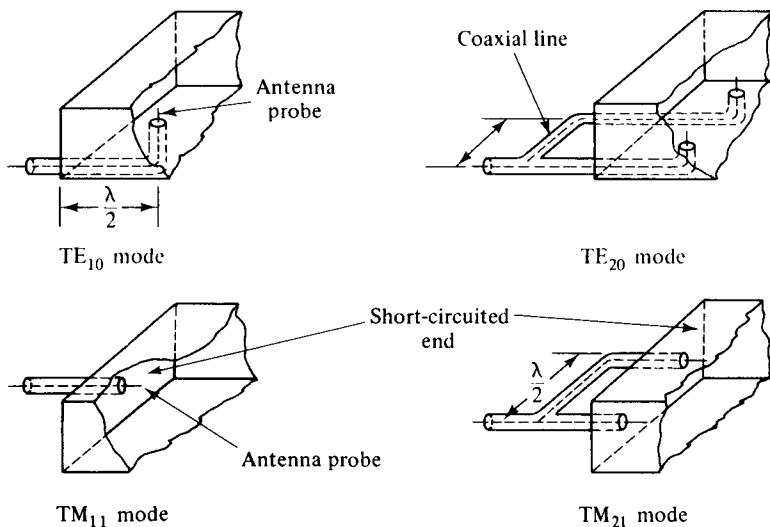
Figure 4-1-6   Methods of exciting various modes in rectangular waveguides.
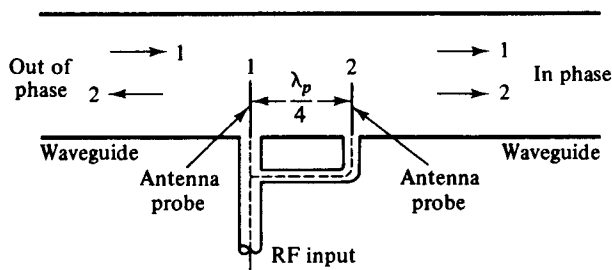

Figure 4-1-7   A method of launching a TE_{10} mode in one direction only.

feeders. The field intensities radiated by the two antennas are in phase opposition to the left of the antennas and cancel each other, whereas in the region to the right of the antennas the field intensities are in time phase and reinforce each other. The resulting wave thus propagates to the right in the guide.

Some higher modes are generated by discontinuities of the waveguide such as obstacles, bends, and loads. However, the higher-order modes are, in general, more highly attenuated than the corresponding dominant mode. On the other hand, the dominant mode tends to remain as a dominant wave even when the guide is large enough to support the higher modes.

4-1-7 Characteristics of Standard Rectangular Waveguides

Rectangular waveguides are commonly used for power transmission at microwave frequencies. Their physical dimensions are regulated by the frequency of the signal being transmitted. For example, at X-band frequencies from 8 to 12 GHz, the outside dimensions of a rectangular waveguide, designated as EIA WR (90) by the Electronic Industry Association, are 2.54 cm (1.0 in.) wide and 1.27 cm (0.5 in.)

TABLE 4-1-7 CHARACTERISTICS OF STANDARD RECTANGULAR WAVEGUIDES

EIA[a] designation WR[b]( )	Physical dimensions				Cutoff frequency for air-filled waveguide in GHz	Recommended frequency range for TE_{10} mode in GHZ
	Inside, in cm (in.)		Outside, in cm (in.)			
	Width	Height	Width	Height		
2300	58.420 (23.000)	29.210 (11.500)	59.055 (23.250)	29.845 (11.750)	0.257	0.32–0.49
2100	53.340 (21.000)	26.670 (10.500)	53.973 (21.250)	27.305 (10.750)	0.281	0.35–0.53
1800	45.720 (18.000)	22.860 (9.000)	46.350 (18.250)	23.495 (9.250)	0.328	0.41–0.62
1500	38.100 (15.000)	19.050 (7.500)	38.735 (15.250)	19.685 (7.750)	0.394	0.49–0.75
1150	29.210 (11.500)	14.605 (5.750)	29.845 (11.750)	15.240 (6.000)	0.514	0.64–0.98
975	24.765 (9.750)	12.383 (4.875)	25.400 (10.000)	13.018 (5.125)	0.606	0.76–1.15
770	19.550 (7.700)	9.779 (3.850)	20.244 (7.970)	10.414 (4.100)	0.767	0.96–1.46
650	16.510 (6.500)	8.255 (3.250)	16.916 (6.660)	8.661 (3.410)	0.909	1.14–1.73
510	12.954 (5.100)	6.477 (2.500)	13.360 (5.260)	6.883 (2.710)	1.158	1.45–2.20
430	10.922 (4.300)	5.461 (2.150)	11.328 (4.460)	5.867 (2.310)	1.373	1.72–2.61
340	8.636 (3.400)	4.318 (1.700)	9.042 (3.560)	4.724 (1.860)	1.737	2.17–3.30
284	7.214 (2.840)	3.404 (1.340)	7.620 (3.000)	3.810 (1.500)	2.079	2.60–3.95
229	5.817 (2.290)	2.908 (1.145)	6.142 (2.418)	3.233 (1.273)	2.579	3.22–4.90
187	4.755 (1.872)	2.215 (0.872)	5.080 (2.000)	2.540 (1.000)	3.155	3.94–5.99
159	4.039 (1.590)	2.019 (0.795)	4.364 (1.718)	2.344 (0.923)	3.714	4.64–7.05
137	3.485 (1.372)	1.580 (0.622)	3.810 (1.500)	1.905 (0.750)	4.304	5.38–8.17
112	2.850 (1.122)	1.262 (0.497)	3.175 (1.250)	1.588 (0.625)	5.263	6.57–9.99
90	2.286 (0.900)	1.016 (0.400)	2.540 (1.000)	1.270 (0.500)	6.562	8.20–12.50
75	1.905 (0.750)	0.953 (0.375)	2.159 (0.850)	1.207 (0.475)	7.874	9.84–15.00
62	1.580 (0.622)	0.790 (0.311)	1.783 (0.702)	0.993 (0.391)	9.494	11.90–18.00
51	1.295 (0.510)	0.648 (0.255)	1.499 (0.590)	0.851 (0.335)	11.583	14.50–22.00
42	1.067 (0.420)	0.432 (0.170)	1.270 (0.500)	0.635 (0.250)	14.058	17.60–26.70

[a] Electronic Industry Association

[b] Rectangular Waveguide

TABLE 4-1-7   CHARACTERISTICS OF STANDARD RECTANGULAR WAVEGUIDES (Cont.)

EIA[a] designation WR[b]( )	Physical dimensions				Cutoff frequency for air-filled waveguide in GHz	Recommended frequency range for TE$_{10}$ mode in GHZ
	Inside, in cm (in.)		Outside, in cm (in.)			
	Width	Height	Width	Height		
34	0.864 (0.340)	0.432 (0.170)	1.067 (0.420)	0.635 (0.250)	17.361	21.70–33.00
28	0.711 (0.280)	0.356 (0.140)	0.914 (0.360)	0.559 (0.220)	21.097	26.40–40.00
22	0.569 (0.224)	0.284 (0.112)	0.772 (0.304)	0.488 (0.192)	26.362	32.90–50.10
19	0.478 (0.188)	0.239 (0.094)	0.681 (0.268)	0.442 (0.174)	31.381	39.20–59.60
15	0.376 (0.148)	0.188 (0.074)	0.579 (0.228)	0.391 (0.154)	39.894	49.80–75.80
12	0.310 (0.122)	0.155 (0.061)	0.513 (0.202)	0.358 (0.141)	48.387	60.50–91.90
10	0.254 (0.100)	0.127 (0.050)	0.457 (0.180)	0.330 (0.130)	59.055	73.80–112.00
8	0.203 (0.080)	0.102 (0.040)	0.406 (0.160)	0.305 (0.120)	73.892	92.20–140.00
7	0.165 (0.065)	0.084 (0.033)	0.343 (0.135)	0.262 (0.103)	90.909	114.00–173.00
5	0.130 (0.051)	0.066 (0.026)	0.257 (0.101)	0.193 (0.076)	115.385	145.00–220.00
4	0.109 (0.043)	0.056 (0.022)	0.211 (0.083)	0.157 (0.062)	137.615	172.00–261.00
3	0.086 (0.034)	0.043 (0.017)	0.163 (0.064)	0.119 (0.047)	174.419	217.00–333.00

high, and its inside dimensions are 2.286 cm (0.90 in.) wide and 1.016 cm (0.40 in.) high. Table 4-1-7 tabulates the characteristics of the standard rectangular waveguides.

4-2 CIRCULAR WAVEGUIDES

A circular waveguide is a tubular, circular conductor. A plane wave propagating through a circular waveguide results in a transverse electric (TE) or transerse magnetic (TM) mode. Several other types of waveguides, such as elliptical and reentrant guides, also propagate electromagnetic waves.

4-2-1 Solutions of Wave Equations in Cylindrical Coordinates

As described in Section 4-1 for rectangular waveguides, only a sinusoidal steady-state or frequency-domain solution will be attempted for circular waveguides. A cylindrical coordinate system is shown in Fig. 4-2-1.
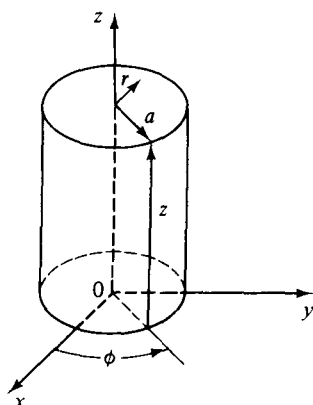
Figure 4-2-1   Cylindrical coordinates.

The scalar Helmholtz equation in cylindrical coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = \gamma^2 \psi \tag{4-2-1}$$

Using the method of separation of variables, the solution is assumed in the form of

$$\Psi = R(r)\Phi(\phi)Z(z) \tag{4-2-2}$$

where $R(r)$ = a function of the r coordinate only
$\Phi(\phi)$ = a function of the ϕ coordinate only
$Z(z)$ = a function of the z coordinate only

Substitution of Eq. (4-2-2) in (4-2-1) and division of the resultant by (4-2-2) yield

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \tag{4-2-3}$$

Since the sum of the three independent terms is a constant, each of the three terms must be a constant. The third term may be set equal to a constant γ_g^2:

$$\frac{d^2Z}{dz^2} = \gamma_g^2 Z \tag{4-2-4}$$

The solutions of this equation are given by

$$Z = Ae^{-\gamma_g z} + Be^{\gamma_g z} \tag{4-2-5}$$

where γ_g = propagation constant of the wave in the guide.
    Inserting γ_g^2 for the third term in the left-hand side of Eq. (4-2-3) and multiplying the resultant by r^2 yield

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} - (\gamma^2 - \gamma_g^2)r^2 = 0 \tag{4-2-6}$$

The second term is a function of ϕ only; hence equating the second term to a con-

stant $(-n^2)$ yields

$$\frac{d^2\Phi}{d\phi^2} = -n^2\Phi \tag{4-2-7}$$

The solution of this equation is also a harmonic function:

$$\Phi = A_n \sin(n\phi) + B_n \cos(n\phi) \tag{4-2-8}$$

Replacing the Φ term by $(-n^2)$ in Eq. (4-2-6) and multiplying through by R, we have

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + [(k_c r)^2 - n^2]R = 0 \tag{4-2-9}$$

This is Bessel's equation of order n in which

$$k_c^2 + \gamma^2 = \gamma_g^2 \tag{4-2-10}$$

This equation is called the *characteristic equation* of Bessel's equation. For a lossless guide, the characteristic equation reduces to

$$\beta_g = \pm\sqrt{\omega^2\mu\epsilon - k_c^2} \tag{4-2-11}$$

The solutions of Bessel's equation are

$$R = C_n J_n(k_c r) + D_n N_n(k_c r) \tag{4-2-12}$$

where $J_n(k_c r)$ is the nth-order Bessel function of the first kind, representing a standing wave of $\cos(k_c r)$ for $r < a$ as shown in Fig. 4-2-2. $N_n(k_c r)$ is the nth-order Bessel function of the second kind, representing a standing wave of $\sin(k_c r)$ for $r > a$ as shown in Fig. 4-2-3.

Therefore the total solution of the Helmholtz equation in cylindrical coordinates is given by

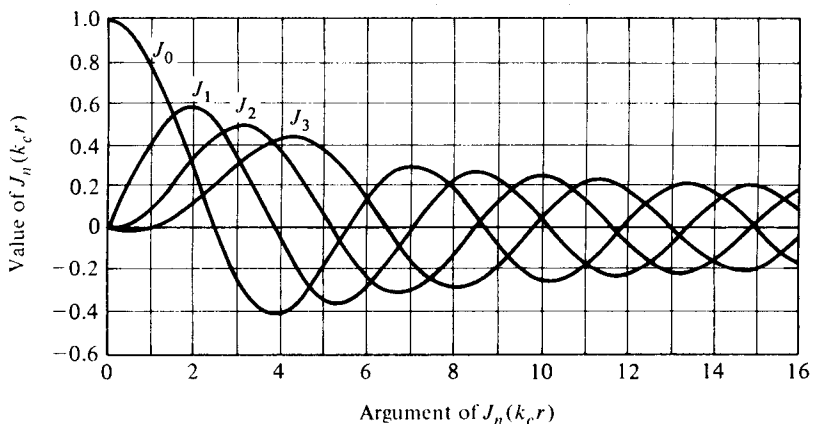$$\Psi = [C_n J_n(k_c r) + D_n N_n(k_c r)][A_n \sin(n\phi) + B_n \cos(n\phi)]e^{\pm j\beta_g z} \tag{4-2-13}$$

Figure 4-2-2   Bessel functions of the first kind.
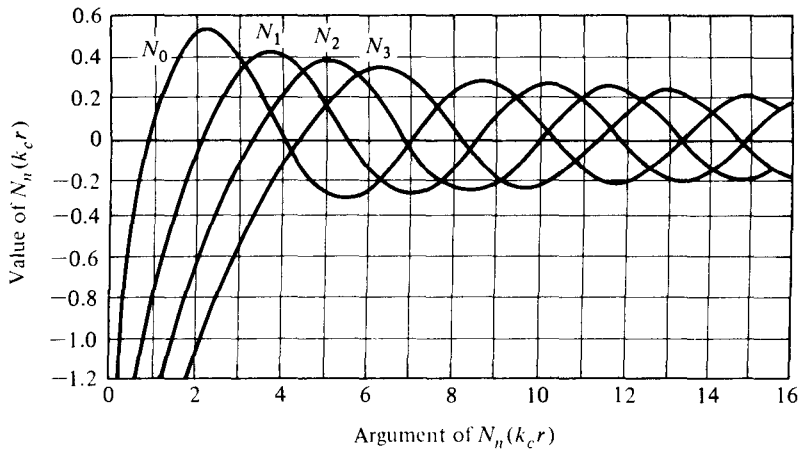
Figure 4-2-3 Bessel functions of the second kind.

At $r = 0$, however, $k_c r = 0$; then the function N_n approaches infinity, so $D_n = 0$. This means that at $r = 0$ on the z axis, the field must be finite. Also, by use of trigonometric manipulations, the two sinusoidal terms become

$$A_n \sin(n\phi) + B_n \cos(n\phi) = \sqrt{A_n^2 + B_n^2} \cos\left[n\phi + \tan^{-1}\left(\frac{A_n}{B_n}\right)\right]$$

$$= F_n \cos(n\phi) \qquad (4\text{-}2\text{-}14)$$

Finally, the solution of the Helmholtz equation is reduced to

$$\Psi = \Psi_0 J_n(k_c r) \cos(n\phi) e^{-j\beta_g z} \qquad (4\text{-}2\text{-}15)$$

4-2-2 TE Modes in Circular Waveguides

It is commonly assumed that the waves in a circular waveguide are propagating in the positive z direction. Figure 4-2-4 shows the coordinates of a circular guide.

The TE_{np} modes in the circular guide are characterized by $E_z = 0$. This means
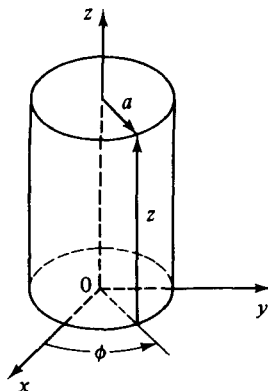

Figure 4-2-4 Coordinates of a circular waveguide.

that the z component of the magnetic field H_z must exist in the guide in order to have electromagnetic energy transmission. A Helmholtz equation for H_z in a circular guide is given by

$$\nabla^2 H_z = \gamma^2 H_z \qquad (4\text{-}2\text{-}16)$$

Its solution is given in Eq. (4-2-15) by

$$H_z = H_{0z} J_n(k_c r) \cos{(n\phi)} e^{-j\beta_g z} \qquad (4\text{-}2\text{-}17)$$

which is subject to the given boundary conditions.

For a lossless dielectric, Maxwell's curl equations in frequency domain are given by

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad (4\text{-}2\text{-}18)$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \qquad (4\text{-}2\text{-}19)$$

In cylindrical coordinates, their components are expressed as

$$\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = -j\omega\mu H_r \qquad (4\text{-}2\text{-}20)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\phi \qquad (4\text{-}2\text{-}21)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial \phi} = -j\omega\mu H_z \qquad (4\text{-}2\text{-}22)$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = j\omega\epsilon E_r \qquad (4\text{-}2\text{-}23)$$

$$-j\beta_g H_r - \frac{\partial H_z}{\partial r} = j\omega\epsilon E_\phi \qquad (4\text{-}2\text{-}24)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} = j\omega\epsilon E_z \qquad (4\text{-}2\text{-}25)$$

When the differentiation $\partial/\partial z$ is replaced by $(-j\beta_g)$ and the z component of electric field E_z by zero, the TE-mode equations in terms of H_z in a circular waveguide are expressed as

$$E_r = -\frac{j\omega\mu}{k_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \phi} \qquad (4\text{-}2\text{-}26)$$

$$E_\phi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial r} \qquad (4\text{-}2\text{-}27)$$

$$E_z = 0 \qquad (4\text{-}2\text{-}28)$$

$$H_r = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial r} \qquad (4\text{-}2\text{-}29)$$

$$H_\phi = \frac{-j\beta_g}{k_c^2} \frac{1}{r} \frac{\partial H_z}{\partial \phi} \tag{4-2-30}$$

$$H_z = H_{0z} J_n(k_c r) \cos(n\phi) e^{-j\beta_g Z} \tag{4-2-31}$$

where $k_c^2 = \omega^2 \mu \epsilon - \beta_g^2$ has been replaced.

The boundary conditions require that the ϕ component of the electric field E_ϕ, which is tangential to the inner surface of the circular waveguide at $r = a$, must vanish or that the r component of the magnetic field H_r, which is normal to the inner surface of $r = a$, must vanish. Consequently

$$E_\phi = 0 \text{ at } r = a \qquad \therefore \left. \frac{\partial H_z}{\partial r} \right|_{r=a} = 0$$

or

$$H_r = 0 \text{ at } r = a \qquad \therefore \left. \frac{\partial H_z}{\partial r} \right|_{r=a} = 0$$

This requirement is equivalent to that expressed in Eq. (4-2-17):

$$\left. \frac{\partial H_z}{\partial r} \right|_{r=a} = H_{0z} J_n'(k_c a) \cos(n\phi) e^{-j\beta_g Z} = 0 \tag{4-2-32}$$

Hence

$$J_n'(k_c a) = 0 \tag{4-2-33}$$

where J_n' indicates the derivative of J_n.

Since the J_n are oscillatory functions, the $J_n'(k_c a)$ are also oscillatory functions. An infinite sequence of values of $(k_c a)$ satisfies Eq. (4-2-32). These points, the roots of Eq. (4-2-32), correspond to the maxima and minima of the curves $J_n'(k_c a)$, as shown in Fig. 4-2-2. Table 4-2-1 tabulates a few roots of $J_n'(k_c a)$ for some lower-order n.

TABLE 4-2-1 pth ZEROS OF $J_n'(K_c a)$ FOR TE$_{np}$ MODES

p	$n =$ 0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	6.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

The permissible values of k_c can be written

$$k_c = \frac{X_{np}'}{a} \tag{4-2-34}$$

Substitution of Eq. (4-2-17) in Eqs. (4-2-26) through (4-2-31) yields the complete field equations of the TE$_{np}$ modes in circular waveguides:

$$E_r = E_{0r} J_n\left(\frac{X_{np}' r}{a}\right) \sin(n\phi) e^{-j\beta_g Z} \tag{4-2-35}$$

$$E_\phi = E_{0\phi} J_n'\left(\frac{X_{np}'r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \qquad (4\text{-}2\text{-}36)$$

$$E_z = 0 \qquad (4\text{-}2\text{-}37)$$

$$H_r = -\frac{E_{0\phi}}{Z_g} J_n\left(\frac{X_{np}'r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \qquad (4\text{-}2\text{-}38)$$

$$H_\phi = \frac{E_{0r}}{Z_g} J_n\left(\frac{X_{np}'r}{a}\right) \sin(n\phi)e^{-j\beta_g z} \qquad (4\text{-}2\text{-}39)$$

$$H_z = H_{0z} J_n\left(\frac{X_{np}'r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \qquad (4\text{-}2\text{-}40)$$

where $Z_g = E_r/H_\phi = -E_\phi/H_r$ has been replaced for the wave impedance in the guide and where $n = 0, 1, 2, 3, \ldots$ and $p = 1, 2, 3, 4, \ldots$.

The first subscript n represents the number of full cycles of field variation in one revolution through 2π rad of ϕ. The second subscript p indicates the number of zeros of E_ϕ—that is, $J_n'(X_{np}'r/a)$ along the radial of a guide, but the zero on the axis is excluded if it exists.

The mode propagation constant is determined by Eqs. (4-2-26) through (4-2-31) and Eq. (4-2-34):

$$\beta_g = \sqrt{\omega^2\mu\epsilon - \left(\frac{X_{np}'}{a}\right)^2} \qquad (4\text{-}2\text{-}41)$$

The cutoff wave number of a mode is that for which the mode propagation constant vanishes. Hence

$$k_c = \frac{X_{np}'}{a} = \omega_c\sqrt{\mu\epsilon} \qquad (4\text{-}2\text{-}42)$$

The cutoff frequency for TE modes in a circular guide is then given by

$$f_c = \frac{X_{np}'}{2\pi a\sqrt{\mu\epsilon}} \qquad (4\text{-}2\text{-}43)$$

and the phase velocity for TE modes is

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \qquad (4\text{-}2\text{-}44)$$

where $v_p = 1/\sqrt{\mu\epsilon} = c/\sqrt{\mu_r\epsilon_r}$ is the phase velocity in an unbounded dielectric.

The wavelength and wave impedance for TE modes in a circular guide are given, respectively, by

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \qquad (4\text{-}2\text{-}45)$$

and

$$Z_g = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \qquad (4\text{-}2\text{-}46)$$

where $\lambda = \dfrac{v_p}{f}$ = wavelength in an unbounded dielectric

$\eta = \sqrt{\dfrac{\mu}{\epsilon}}$ = intrinsic impedance in an unbounded dielectric

---

Example 4-2-1:   TE Mode in Circular Waveguide

A TE_{11} mode is propagating through a circular waveguide. The radius of the guide is 5 cm, and the guide contains an air dielectric (refer to Fig. 4-2-5).
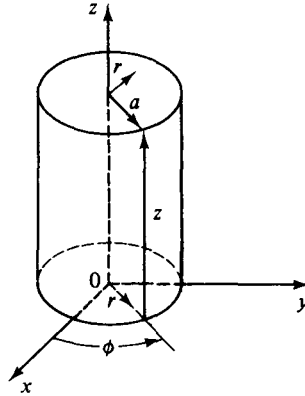

Figure 4-2-5   Diagram for Example 4-2-1.

a. Determine the cutoff frequency.
b. Determine the wavelength λ_g in the guide for an operating frequency of 3 GHz.
c. Determine the wave impedance Z_g in the guide.

Solution

a. From Table 4-2-1 for TE_{11} mode, $n = 1$, $p = 1$, and $X'_{11} = 1.841 = k_c a$. The cutoff wave number is

$$k_c = \frac{1.841}{a} = \frac{1.841}{5 \times 10^{-2}} = 36.82$$

The cutoff frequency is

$$f_c = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}} = \frac{(36.82)(3 \times 10^8)}{2\pi} = 1.758 \times 10^9 \text{ Hz}$$

b. The phase constant in the guide is

$$\beta_g = \sqrt{\omega^2\mu_0\epsilon_0 - k_c^2}$$

$$= \sqrt{(2\pi \times 3 \times 10^9)^2(4\pi \times 10^{-7} \times 8.85 \times 10^{-12}) - (36.82)^2}$$

$$= 50.9 \text{ rads/m}$$

The wavelength in the guide is

$$\lambda_g = \frac{2\pi}{\beta_g} = \frac{6.28}{50.9} = 12.3 \text{ cm}$$

c. The wave impedance in the guide is

$$Z_g = \frac{\omega\mu_0}{\beta_g} = \frac{(2\pi \times 3 \times 10^9)(4\pi \times 10^{-7})}{50.9} = 465 \ \Omega$$

4-2-3 TM Modes in Circular Waveguides

The TM_{np} modes in a circular guide are characterized by $H_z = 0$. However, the z component of the electric field E_z must exist in order to have energy transmission in the guide. Consequently, the Helmholtz equation for E_z in a circular waveguide is given by

$$\nabla^2 E_z = \gamma^2 E_z \tag{4-2-47}$$

Its solution is given in Eq. (4-2-15) by

$$E_z = E_{0z} J_n(k_c r) \cos (n\phi) e^{-j\beta_g z} \tag{4-2-48}$$

which is subject to the given boundary conditions.

The boundary condition requires that the tangential component of electric field E_z at $r = a$ vanishes. Consequently,

$$J_n(k_c a) = 0 \tag{4-2-49}$$

Since $J_n(k_c r)$ are oscillatory functions, as shown in Fig. 4-2-2, there are infinite numbers of roots of $J_n(k_c r)$. Table 4-2-2 tabulates a few of them for some lower-order n.

TABLE 4-2-2 pth ZEROS OF $J_n(K_c a)$ FOR TM_{np} MODES

p	$n =$ 0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.106	8.417	9.761	11.065	12.339
3	8.645	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

For $H_z = 0$ and $\partial/\partial z = -j\beta_g$, the field equations in the circular guide, after expanding $\nabla \times E = -j\omega\mu H$ and $\nabla \times H = j\omega\epsilon E$, are given by

$$E_r = \frac{-j\beta_g}{k_c^2} \frac{\partial E_z}{\partial r} \tag{4-2-50}$$

$$E_\phi = \frac{-j\beta_g}{k_c^2} \frac{1}{r} \frac{\partial E_z}{\partial \phi} \tag{4-2-51}$$

$$E_z = \text{Eq. (4-2-48)} \tag{4-2-52}$$

$$H_r = \frac{j\omega\epsilon}{k_c^2} \frac{1}{r} \frac{\partial E_z}{\partial \phi} \tag{4-2-53}$$

$$H_\phi = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial r} \tag{4-2-54}$$

$$H_z = 0 \tag{4-2-55}$$

where $k_c^2 = \omega^2\mu\epsilon - \beta_g^2$ has been replaced.

Differentiation of Eq. (4-2-48) with respect to z and substitution of the result in Eqs. (4-2-50) through (4-2-55) yield the field equations of TM_{np} modes in a circular waveguide:

$$E_r = E_{0r}J_n'\left(\frac{X_{np}r}{a}\right)\cos{(n\phi)}e^{-j\beta_g z} \tag{4-2-56}$$

$$E_\phi = E_{0\phi}J_n\left(\frac{X_{np}r}{a}\right)\sin{(n\phi)}e^{-j\beta_g z} \tag{4-2-57}$$

$$E_z = E_{0z}J_n\left(\frac{X_{np}r}{a}\right)\cos{(n\phi)}e^{-j\beta_g z} \tag{4-2-58}$$

$$H_r = \frac{E_{0\phi}}{Z_g}J_n\left(\frac{X_{np}r}{a}\right)\sin{(n\phi)}e^{-j\beta_g z} \tag{4-2-59}$$

$$H_\phi = \frac{E_{0r}}{Z_g}J_n'\left(\frac{X_{np}r}{a}\right)\cos{(n\phi)}e^{-j\beta_g z} \tag{4-2-60}$$

$$H_z = 0 \tag{4-2-61}$$

where $Z_g = E_r/H_\phi = -E_\phi/H_r = \beta_g/(\omega\epsilon)$ and $k_c = X_{np}/a$ have been replaced and where $n = 0, 1, 2, 3, \ldots$ and $p = 1, 2, 3, 4, \ldots$.

Some of the TM-mode characteristic equations in the circular guide are identical to those of the TE mode, but some are different. For convenience, all are shown here:

$$\beta_g = \sqrt{\omega^2\mu\epsilon - \left(\frac{X_{np}}{a}\right)^2} \tag{4-2-62}$$

$$k_c = \frac{X_{np}}{a} = \omega_c\sqrt{\mu\epsilon} \tag{4-2-63}$$

$$f_c = \frac{X_{np}}{2\pi a\sqrt{\mu\epsilon}} \tag{4-2-64}$$

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{4-2-65}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{4-2-66}$$

$$Z_g = \frac{B_g}{\omega\epsilon} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-2-67}$$

It should be noted that the dominant mode, or the mode of lowest cutoff frequency

in a circular waveguide, is the mode of TE_{11} that has the smallest value of the product, $k_c a = 1.841$, as shown in Tables 4-2-1 and 4-2-2.

Example 4-2-2:   Wave Propagation in Circular Waveguide

An air-filled circular waveguide has a radius of 2 cm and is to carry energy at a frequency of 10 GHz. Find all the TE_{np} and TM_{np} modes for which energy transmission is possible.

Solution   Since the physical dimension of the guide and the frequency of the wave remain constant, the product of $(k_c a)$ is also constant. Thus

$$k_c a = (\omega_c \sqrt{\mu_0 \epsilon_0})a = \frac{2\pi \times 10^{10}}{3 \times 10^8}(2 \times 10^{-2}) = 4.18$$

Any mode having a product of $(k_c a)$ less than or equal to 4.18 will propagate the wave with a frequency of 10 GHz. This is

$$k_c a \leqq 4.18$$

The possible modes are

$TE_{11}(1.841)$	$TM_{01}(2.405)$
$TE_{21}(3.054)$	$TM_{11}(3.832)$
$TE_{01}(3.832)$	

4-2-4 TEM Modes in Circular Waveguides

The transverse electric and transverse magnetic (TEM) modes or transmission-line modes are characterized by

$$E_z = H_z = 0$$

This means that the electric and magnetic fields are completely transverse to the direction of wave propagation. This mode cannot exist in hollow waveguides, since it requires two conductors, such as the coaxial transmission line and two-open-wire line. Analysis of the TEM mode illustrates an excellent analogous relationship between the method of circuit theory and that of the field theory. Figure 4-2-6 shows a coaxial line.

Maxwell's curl equations in cylindrical coordinates

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{4-2-68}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \tag{4-2-69}$$

become

$$B_g E_r = \omega\mu H_\phi \tag{4-2-70}$$
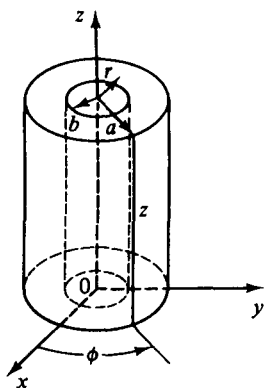
$$B_g E_\phi = \omega\mu H_r \tag{4-2-71}$$

Figure 4-2-6   Coordinates of a coaxial line.

$$\frac{\partial}{\partial r}(rE_\phi) - \frac{\partial Er}{\partial \phi} = 0 \qquad (4\text{-}2\text{-}72)$$

$$\beta_g H_r = -\omega\epsilon E_\phi \qquad (4\text{-}2\text{-}73)$$

$$\beta_g H_\phi = \omega\epsilon E_r \qquad (4\text{-}2\text{-}74)$$

$$\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial \phi} = 0 \qquad (4\text{-}2\text{-}75)$$

where $\partial/\partial r = -j\beta_g$ and $E_z = H_z = 0$ are replaced.

Substitution of Eq. (4-2-71) in (4-2-73) yields the propagation constant of the TEM mode in a coaxial line:

$$\beta_g = \omega\sqrt{\mu\epsilon} \qquad (4\text{-}2\text{-}76)$$

which is the phase constant of the wave in a lossless transmission line with a dielectric.

In comparing the preceding equation with the characteristic equation of the Helmholtz equation in cylindrical coordinates as given in Eq. (4-2-11) by

$$\beta_g = \sqrt{\omega^2\mu\epsilon - k_c^2} \qquad (4\text{-}2\text{-}77)$$

it is evident that

$$k_c = 0 \qquad (4\text{-}2\text{-}78)$$

This means that the cutoff frequency of the TEM mode in a coaxial line is zero, which is the same as in ordinary transmission lines.

The phase velocity of the TEM mode can be expressed from Eq. (4-2-76) as

$$v_p = \frac{\omega}{\beta_g} = \frac{1}{\sqrt{\mu\epsilon}} \qquad (4\text{-}2\text{-}79)$$

which is the velocity of light in an unbounded dielectric.

The wave impedance of the TEM mode is found from either Eqs. (4-2-70) and (4-2-73) or Eqs. (4-2-71) and (4-2-74) as

$$\eta\,(\text{TEM}) \;=\; \sqrt{\frac{\mu}{\epsilon}} \qquad\qquad (4\text{-}2\text{-}80)$$

which is the wave impedance of a lossless transmission line in a dielectric.

Ampère's law states that the line integral of **H** about any closed path is exactly equal to the current enclosed by that path. This is

$$\oint \mathbf{H} \cdot d\ell \;=\; I \;=\; I_0 e^{-j\beta_g z} \;=\; 2\pi r H_\phi \qquad\qquad (4\text{-}2\text{-}81)$$

where I is the complex current that must be supported by the center conductor of a coaxial line. This clearly demonstrates that the TEM mode can only exist in the two-conductor system—not in the hollow waveguide because the center conductor does not exist.

In summary, the properties of TEM modes in a lossless medium are as follows:

1. Its cutoff frequency is zero.
2. Its transmission line is a two-conductor system.
3. Its wave impedance is the impedance in an unbounded dielectric.
4. Its propagation constant is the constant in an unbounded dielectric.
5. Its phase velocity is the velocity of light in an unbounded dielectric.

4-2-5 Power Transmission in Circular Waveguides or Coaxial Lines

In general, the power transmitted through circular waveguides and coaxial lines can be calculated by means of the complex Poynting theorem described in Section 2-2. For a lossless dielectric, the time-average power transmitted through a circular guide can be given by

$$P_{\text{tr}} \;=\; \frac{1}{2Z_g} \int_0^{2\pi}\!\!\int_0^a [|E_\phi|^2 + |E_\phi|^2] r\,dr\,d\phi \qquad\qquad (4\text{-}2\text{-}82)$$

$$P_{\text{tr}} \;=\; \frac{Z_g}{2} \int_0^{2\pi}\!\!\int_0^a [|H_r|^2 + |H_\phi|^2] r\,dr\,d\phi \qquad\qquad (4\text{-}2\text{-}83)$$

where $Z_g = \dfrac{E_r}{H_\phi} = -\dfrac{E_\phi}{H_r} = $ wave impedance in the guide

$\qquad a = $ radius of the circular guide

Substitution of Z_g for a particular mode in Eq. (4-2-82) yields the power transmitted by that mode through the guide.

For TE$_{np}$ modes, the average power transmitted through a circular guide is given by

$$P_{tr} = \frac{\sqrt{1 - (f_c/f)^2}}{2\eta} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \qquad (4\text{-}2\text{-}84)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

For TM$_{np}$ modes, the average power transmitted through a circular guide is given by

$$P_{tr} = \frac{1}{2\eta \sqrt{1 - (f_c/f)^2}} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \qquad (4\text{-}2\text{-}85)$$

For TEM modes in coaxial lines, the average power transmitted through a coaxial line or two-open-wire line is given by

$$P_{tr} = \frac{1}{2\eta} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \qquad (4\text{-}2\text{-}86)$$

If the current carried by the center conductor of a coaxial line is assumed to be

$$I_z = I_0 e^{-j\beta_g Z} \qquad (4\text{-}2\text{-}87)$$

the magnetic intensity induced by the current around the center conductor is given by Ampère's law as

$$H_\phi = \frac{I_0}{2\pi r} e^{-j\beta_g Z} \qquad (4\text{-}2\text{-}88)$$

The potential rise from the outer conductor to the center conductor is given by

$$V_r = -\int_b^a E_r dr = -\int_b^a \eta H_\phi dr = \frac{I_0 \eta}{2\pi} \ln\left(\frac{b}{a}\right) e^{-j\beta_g Z} \qquad (4\text{-}2\text{-}89)$$

The characteristic impedance of a coaxial line is

$$Z_0 = \frac{V}{I} = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right) \qquad (4\text{-}2\text{-}90)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

The power transmitted by TEM modes in a coaxial line can be expressed from Eq. (4-2-86) as

$$P_{tr} = \frac{1}{2\eta} \int_0^{2\pi} \int_a^b |\eta H_\phi|^2 \, r \, dr \, d\phi = \frac{\eta I_0^2}{4\pi} \ln\left(\frac{b}{a}\right) \qquad (4\text{-}2\text{-}91)$$

Substitution of $|V_r|$ from Eq. (4-2-89) into Eq. (4-2-91) yields

$$P_{tr} = \tfrac{1}{2} V_0 I_0 \qquad (4\text{-}2\text{-}92)$$

This shows that the power transmission derived from the Poynting theory is the same as from the circuit theory for an ordinary transmission line.

4-2-6 Power Losses in Circular Waveguides or Coaxial Lines

The theory and equations derived in Section 4-1-5 for TE and TM modes in rectangular waveguides are applicable to TE and TM modes in circular guides. The power losses for the TEM mode in coaxial lines can be computed from transmission-line theory by means of

$$P_L = 2\alpha P_{tr} \tag{4-2-93}$$

where P_L = power loss per unit length
P_{tr} = transmitted power
α = attenuation constant

For a low-loss conductor, the attenuation constant is given by

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}} \right) \tag{4-2-94}$$

4-2-7 Excitations of Modes in Circular Waveguides

As described earlier, TE modes have no z component of an electric field, and TM modes have no z component of magnetic intensity. If a device is inserted in a circular guide in such a way that it excites only a z component of electric intensity, the wave propagating through the guide will be the TM mode; on the other hand, if a device is placed in a circular guide in such a manner that only the z component of magnetic intensity exists, the traveling wave will be the TE mode. The methods of excitation for various modes in circular waveguides are shown in Fig. 4-2-7.
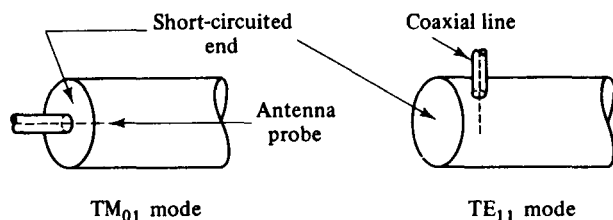
Figure 4-2-7   Methods of exciting various modes in circular waveguides.

A common way to excite TM modes in a circular guide is by a coaxial line as shown in Fig. 4-2-8. At the end of the coaxial line a large magnetic intensity exists in the ϕ direction of wave propagation. The magnetic field from the coaxial line will excite the TM modes in the guide. However, when the guide is connected to the source by a coaxial line, a discontinuity problem at the junction will increase the standing-wave ratio on the line and eventually decrease the power transmission. It is often necessary to place a turning device around the junction in order to suppress the reflection.
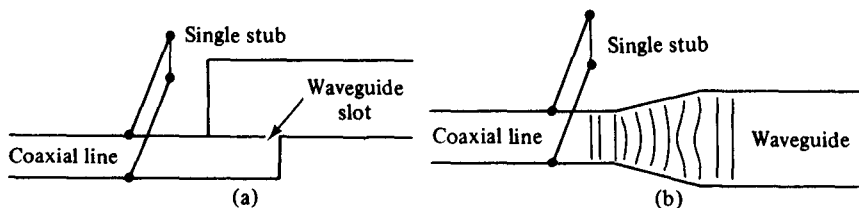
Figure 4-2-8 Methods of exciting TM modes in a circular waveguide. (a) Coaxial line with a slotted waveguide. (b) Coaxial line in series with a circular waveguide.

TABLE 4-2-8 CHARACTERISTICS OF STANDARD CIRCULAR WAVEGUIDES

EIA[a] designation WC[b]( )	Inside diameter in cm (in.)	Cutoff frequency for air-filled waveguide in GHz	Recommended frequency range for TE$_{11}$ mode in GHz
992	25.184 (9.915)	0.698	0.80–1.10
847	21.514 (8.470)	0.817	0.94–1.29
724	18.377 (7.235)	0.957	1.10–1.51
618	15.700 (6.181)	1.120	1.29–1.76
528	13.411 (5.280)	1.311	1.51–2.07
451	11.458 (4.511)	1.534	1.76–2.42
385	9.787 (3.853)	1.796	2.07–2.83
329	8.362 (3.292)	2.102	2.42–3.31
281	7.142 (2.812)	2.461	2.83–3.88
240	6.104 (2.403)	2.880	3.31–4.54
205	5.199 (2.047)	3.381	3.89–5.33
175	4.445 (1.750)	3.955	4.54–6.23
150	3.810 (1.500)	4.614	5.30–7.27
128	3.254 (1.281)	5.402	6.21–8.51
109	2.779 (1.094)	6.326	7.27–9.97
94	2.383 (0.938)	7.377	8.49–11.60
80	2.024 (0.797)	8.685	9.97–13.70
69	1.748 (0.688)	10.057	11.60–15.90
59	1.509 (0.594)	11.649	13.40–18.40
50	1.270 (0.500)	13.842	15.90–21.80
44	1.113 (0.438)	15.794	18.20–24.90
38	0.953 (0.375)	18.446	21.20–29.10
33	0.833 (0.328)	21.103	24.30–33.20
28	0.714 (0.281)	24.620	28.30–38.80
25	0.635 (0.250)	27.683	31.80–43.60
22	0.556 (0.219)	31.617	36.40–49.80
19	0.478 (0.188)	36.776	42.40–58.10
17	0.437 (0.172)	40.227	46.30–63.50
14	0.358 (0.141)	49.103	56.60–77.50
13	0.318 (0.125)	55.280	63.50–87.20
11	0.277 (0.109)	63.462	72.70–99.70
9	0.239 (0.094)	73.552	84.80–116.00

[a] Electronic Industry Association
[b] Circular Waveguide

4-2-8 Characteristics of Standard Circular Waveguides

The inner diameter of a circular waveguide is regulated by the frequency of the signal being transmitted. For example: at X-band frequencies from 8 to 12 GHz, the inner diameter of a circular waveguide designated as EIA WC(94) by the Electronic Industry Association is 2.383 cm (0.938 in.). Table 4-2-8 tabulates the characteristics of the standard circular waveguides.

4-3 MICROWAVE CAVITIES

In general, a cavity resonator is a metallic enclosure that confines the electromagnetic energy. The stored electric and magnetic energies inside the cavity determine its equivalent inductance and capacitance. The energy dissipated by the finite conductivity of the cavity walls determines its equivalent resistance. In practice, the rectangular-cavity resonator, circular-cavity resonator, and reentrant-cavity resonator are commonly used in many microwave applications.

Theoretically a given resonator has an infinite number of resonant modes, and each mode corresponds to a definite resonant frequency. When the frequency of an impressed signal is equal to a resonant frequency, a maximum amplitude of the standing wave occurs, and the peak energies stored in the electric and magnetic fields are equal. The mode having the lowest resonant frequency is known as the *dominant mode*.

4-3-1 Rectangular-Cavity Resonator

The electromagnetic field inside the cavity should satisfy Maxwell's equations, subject to the boundary conditions that the electric field tangential to and the magnetic field normal to the metal walls must vanish. The geometry of a rectangular cavity is illustrated in Fig. 4-3-1.

The wave equations in the rectangular resonator should satisfy the boundary condition of the zero tangential E at four of the walls. It is merely necessary to choose the harmonic functions in z to satisfy this condition at the remaining two end
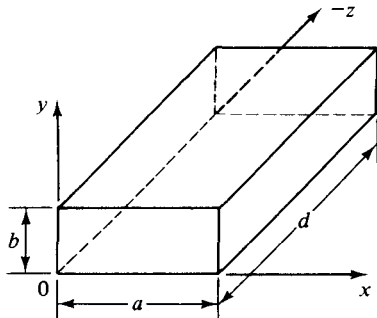

Figure 4-3-1 Coordinates of a rectangular cavity.

walls. These functions can be found if

$$H_z = H_{0z} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right) \qquad (\text{TE}_{mnp}) \qquad (4\text{-}3\text{-}1)$$

where $m = 0, 1, 2, 3, \ldots$ represents the number of the half-wave periodicity in the x direction

$n = 0, 1, 2, 3, \ldots$ represents the number of the half-wave periodicity in the y direction

$p = 1, 2, 3, 4, \ldots$ represents the number of the half-wave periodicity in the z direction

and

$$E_z = E_{0z} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{b}\right) \cos\left(\frac{p\pi z}{d}\right) \qquad (\text{TM}_{mnp}) \qquad (4\text{-}3\text{-}2)$$

where $m = 1, 2, 3, 4, \ldots$
$n = 1, 2, 3, 4, \ldots$
$p = 0, 1, 2, 3, \ldots$

The separation equation for both TE and TM modes is given by

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \qquad (4\text{-}3\text{-}3)$$

For a lossless dielectric, $k^2 = \omega^2 \mu\epsilon$; therefore, the resonant frequency is expressed by

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \qquad (\text{TE}_{mnp}, \text{TM}_{mnp}) \qquad (4\text{-}3\text{-}4)$$

For $a > b < d$, the dominant mode is the TE$_{101}$ mode.

In general, a straight-wire probe inserted at the position of maximum electric intensity is used to excite a desired mode, and the loop coupling placed at the position of maximum magnetic intensity is utilized to launch a specific mode. Figure 4-3-2 shows the methods of excitation for the rectangular resonator. The maximum amplitude of the standing wave occurs when the frequency of the impressed signal is equal to the resonant frequency.
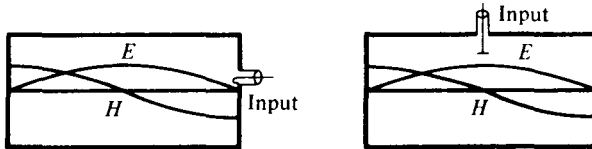

Figure 4-3-2 Methods of exciting wave modes in a resonator.

4-3-2 Circular-Cavity Resonator and Semicircular-Cavity Resonator

Circular-cavity resonator. A circular-cavity resonator is a circular wave-guide with two ends closed by a metal wall (see Fig. 4-3-3). The wave function in the circular resonator should satisfy Maxwell's equations, subject to the same
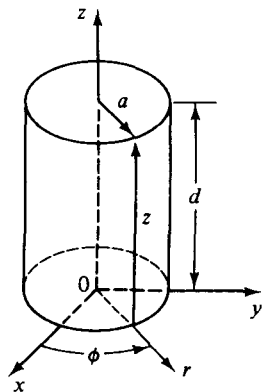
Figure 4-3-3 Coordinates of a circular resonator.

boundary conditions described for a rectangular-cavity resonator. It is merely necessary to choose the harmonic functions in z to satisfy the boundary conditions at the remaining two end walls. These can be achieved if

$$H_z = H_{0z} J_n\left(\frac{X'_{np}r}{a}\right) \cos{(n\phi)} \sin{\left(\frac{q\pi z}{d}\right)} \qquad (\text{TE}_{npq}) \qquad (4\text{-}3\text{-}5)$$

where $n = 0, 1, 2, 3, \ldots$ is the number of the periodicity in the ϕ direction
$\qquad p = 1, 2, 3, 4, \ldots$ is the number of zeros of the field in the radial direction
$\qquad q = 1, 2, 3, 4, \ldots$ is the number of half-waves in the axial direction
$\qquad J_n = $ Bessell's function of the first kind
$\qquad H_{0z} = $ amplitude of the magnetic field

and

$$E_z = E_{0z} J_n\left(\frac{X_{np}r}{a}\right) \cos{(n\phi)} \cos{\left(\frac{q\pi z}{d}\right)} \qquad (\text{TM}_{npq}) \qquad (4\text{-}3\text{-}6)$$

where $n = 0, 1, 2, 3, \ldots$
$\qquad p = 1, 2, 3, 4, \ldots$
$\qquad q = 0, 1, 2, 3, \ldots$
$\qquad E_{0z} = $ amplitude of the electric field

The separation equations for TE and TM modes are given by

$$k^2 = \left(\frac{X'_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 \qquad (\text{TE mode}) \qquad (4\text{-}3\text{-}7)$$

$$k^2 = \left(\frac{X_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 \qquad (\text{TM mode}) \qquad (4\text{-}3\text{-}8)$$

Substitution of $k^2 = \omega^2 \mu\epsilon$ in Eqs. (4-3-7) and (4-3-8) yields the resonant frequencies for TE and TM modes, respectively, as

$$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{X'_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2} \qquad (\text{TE}) \qquad (4\text{-}3\text{-}9)$$

$$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{X_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2} \qquad \text{(TM)} \qquad \text{(4-3-10)}$$

It is interesting to note that the TM_{110} mode is dominant where $2a > d$, and that the TE_{111} mode is dominant when $d \geq 2a$.

Semicircular-cavity resonator.   A semicircular-cavity resonator is shown in Fig. 4-3-4. The wave function of the TE_{npq} mode in the semicircular resonator can be written

$$H_z = H_{0z} J_n\left(\frac{X'_{np}r}{a}\right) \cos(n\phi) \sin\left(\frac{q\pi z}{d}\right) \qquad \text{(TE mode)} \qquad \text{(4-3-11)}$$

where $n = 0, 1, 2, 3, \ldots$
$\quad p = 1, 2, 3, 4, \ldots$
$\quad q = 1, 2, 3, 4, \ldots$
$\quad a$ = radius of the semicircular-cavity resonator
$\quad d$ = length of the resonator
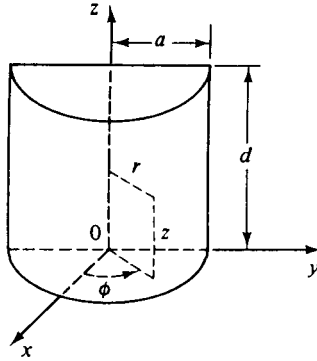
Figure 4-3-4   Semicircular resonator.

The wave function of the TM_{npq} mode in the semicircular-cavity resonator can be written

$$E_z = E_{02} J_n\left(\frac{X_{np}r}{a}\right) \sin(n\phi) \cos\left(\frac{q\pi}{d}z\right) \qquad \text{(TM mode)} \qquad \text{(4-3-12)}$$

where $n = 1, 2, 3, 4, \ldots$
$\quad p = 1, 2, 3, 4, \ldots$
$\quad q = 0, 1, 2, 3, \ldots$

With the separation equations given in Eqs. (4-3-7) and (4-3-8), the equations of resonant frequency for TE and TM modes in a semicircular-cavity resonator are the same as in the circular-cavity resonator. They are repeated as follows:

$$f_r = \frac{1}{2\pi a\sqrt{\mu\epsilon}}\sqrt{(X'_{np})^2 + \left(\frac{q\pi a}{d}\right)^2} \qquad (TE_{npq} \text{ mode}) \qquad \text{(4-3-13)}$$

$$f_r = \frac{1}{2\pi a \sqrt{\mu \epsilon}} \sqrt{(X_{np})^2 + \left(\frac{q\pi a}{d}\right)^2} \qquad (\text{TM}_{npq} \text{ mode}) \qquad (4\text{-}3\text{-}14)$$

However, the values of the subscripts n, p, and q differ from those for the circular-cavity resonator. Also, it must be emphasized that the TE_{111} mode is dominant if $d > a$ and that the TM_{110} mode is dominant if $d < a$.

4-3-3 Q Factor of a Cavity Resonator

The quality factor Q is a measure of the frequency selectivity of a resonant or antiresonant circuit, and it is defined as

$$Q \equiv 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}} = \frac{\omega W}{P} \qquad (4\text{-}3\text{-}15)$$

where W is the maximum stored energy and P is the average power loss.

At resonant frequency, the electric and magnetic energies are equal and in time quadrature. When the electric energy is maximum, the magnetic energy is zero and vice versa. The total energy stored in the resonator is obtained by integrating the energy density over the volume of the resonator:

$$W_e = \int_v \frac{\epsilon}{2} |E|^2 \, dv = W_m = \int_v \frac{\mu}{2} |H|^2 \, dv = W \qquad (4\text{-}3\text{-}16)$$

where $|E|$ and $|H|$ are the peak values of the field intensities.

The average power loss in the resonator can be evaluated by integrating the power density as given in Eq. (2-5-12) over the inner surface of the resonator. Hence

$$P = \frac{R_s}{2} \int_s |H_t|^2 \, da \qquad (4\text{-}3\text{-}17)$$

where H_t is the peak value of the tangential magnetic intensity and R_s is the surface resistance of the resonator.

Substitution of Eqs. (4-3-16) and (4-3-17) in Eq. (4-3-15) yields

$$Q = \frac{\omega \mu \int_v |H|^2 \, dv}{R_s \int_s |H_t|^2 \, da} \qquad (4\text{-}3\text{-}18)$$

Since the peak value of the magnetic intensity is related to its tangential and normal components by

$$|H|^2 = |H_t|^2 + |H_n|^2$$

where H_n is the peak value of the normal magnetic intensity, the value of $|H_t|^2$ at the resonator walls is approximately twice the value of $|H|^2$ averaged over the volume. So the Q of a cavity resonator as shown in Eq. (4-3-18) can be expressed approximately by

$$Q = \frac{\omega \mu (\text{volume})}{2R_s(\text{surface areas})} \qquad (4\text{-}3\text{-}19)$$

An unloaded resonator can be represented by either a series or a parallel resonant circuit. The resonant frequency and the unloaded Q_0 of a cavity resonator are

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{4-3-20}$$

$$Q_0 = \frac{\omega_0 L}{R} \tag{4-3-21}$$

If the cavity is coupled by means of an ideal $N{:}1$ transformer and a series inductance L_s to a generator having internal impedance Z_g, then the coupling circuit and its equivalent are as shown in Fig. 4-3-5.
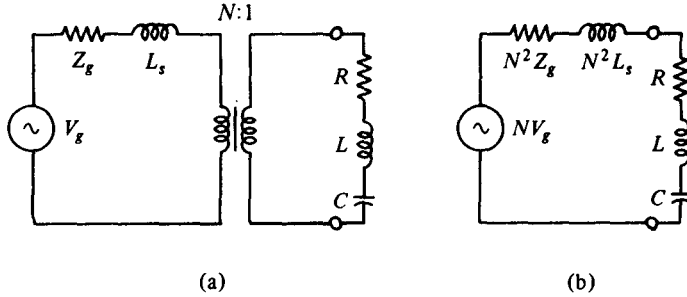

Figure 4-3-5 Cavity coupled to a generator. (a) Coupling circuit. (b) Equivalent circuit.

The loaded Q_ℓ of the system is given by

$$Q_\ell = \frac{\omega_0 L}{R + N^2 Z_g} \qquad \text{for } |N^2 L_s| \ll |R + N^2 Z_g| \tag{4-3-22}$$

The coupling coefficient of the system is defined as

$$K \equiv \frac{N^2 Z_g}{R} \tag{4-3-23}$$

and the loaded Q_ℓ would become

$$Q_\ell = \frac{\omega_0 L}{R(1 + K)} = \frac{Q_0}{1 + K} \tag{4-3-24}$$

Rearrangement of Eq. (4-3-24) yields

$$\frac{1}{Q_\ell} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} \tag{4-3-25}$$

where $Q_{\text{ext}} = Q_0/K = \omega_0 L/(KR)$ is the external Q.

There are three types of coupling coefficients:

1. Critical coupling: If the resonator is matched to the generator, then

$$K = 1 \tag{4-3-26}$$

The loaded Q_ℓ is given by

$$Q_\ell = \tfrac{1}{2}Q_{\text{ext}} = \tfrac{1}{2}Q_0 \qquad (4\text{-}3\text{-}27)$$

2. **Overcoupling:** If $K > 1$, the cavity terminals are at a voltage maximum in the input line at resonance. The normalized impedance at the voltage maximum is the standing-wave ratio ρ. That is

$$K = \rho \qquad (4\text{-}3\text{-}28)$$

The loaded Q_ℓ is given by

$$Q_\ell = \frac{Q_0}{1 + \rho} \qquad (4\text{-}3\text{-}29)$$

3. **Undercoupling:** If $K < 1$, the cavity terminals are at a voltage minimum and the input terminal impedance is equal to the reciprocal of the standing-wave ratio. That is,

$$K = \frac{1}{\rho} \qquad (4\text{-}3\text{-}30)$$

The loaded Q_ℓ is given by

$$Q_\ell = \frac{\rho}{\rho + 1}Q_0 \qquad (4\text{-}3\text{-}31)$$

The relationship of the coupling coefficient K and the standing-wave ratio is shown in Fig. 4-3-6.
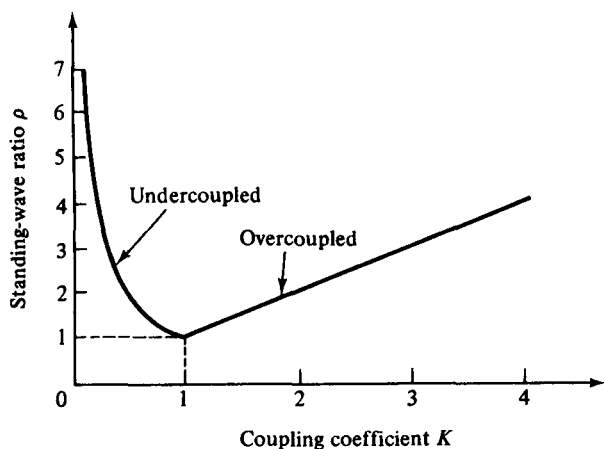
Figure 4-3-6 Coupling coefficient versus standing-wave ratio.

4-4 MICROWAVE HYBRID CIRCUITS

A microwave circuit ordinarily consists of several microwave devices connected in some way to achieve the desired transmission of a microwave signal. The interconnection of two or more microwave devices may be regarded as a microwave junction. Commonly used microwave junctions include such waveguide tees as the *E*-

plane tee, H-plane tee, magic tee, hybrid ring (rat-race circuit), directional coupler, and the circulator. This section describes these microwave hybrids, which are shown in Fig. 4-4-1.

A two-port network is shown in Fig. 4-4-2. From network theory a two-port device can be described by a number of parameter sets, such as the H, Y, Z, and $ABCD$ parameters.

H parameters: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$ $V_1 = h_{11}I_1 + h_{12}V_2$ (4-4-1)

$I_2 = h_{21}I_1 + h_{22}V_2$ (4-4-2)

Y parameters: $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ $I_1 = y_{11}V_1 + y_{12}V_2$ (4-4-3)

$I_2 = y_{21}V_1 + y_{22}V_1$ (4-4-4)

(a) E-plane tee

(b) H-plane tee

(c) Magic tee

(d) Hybrid ring

(e) Directional coupler
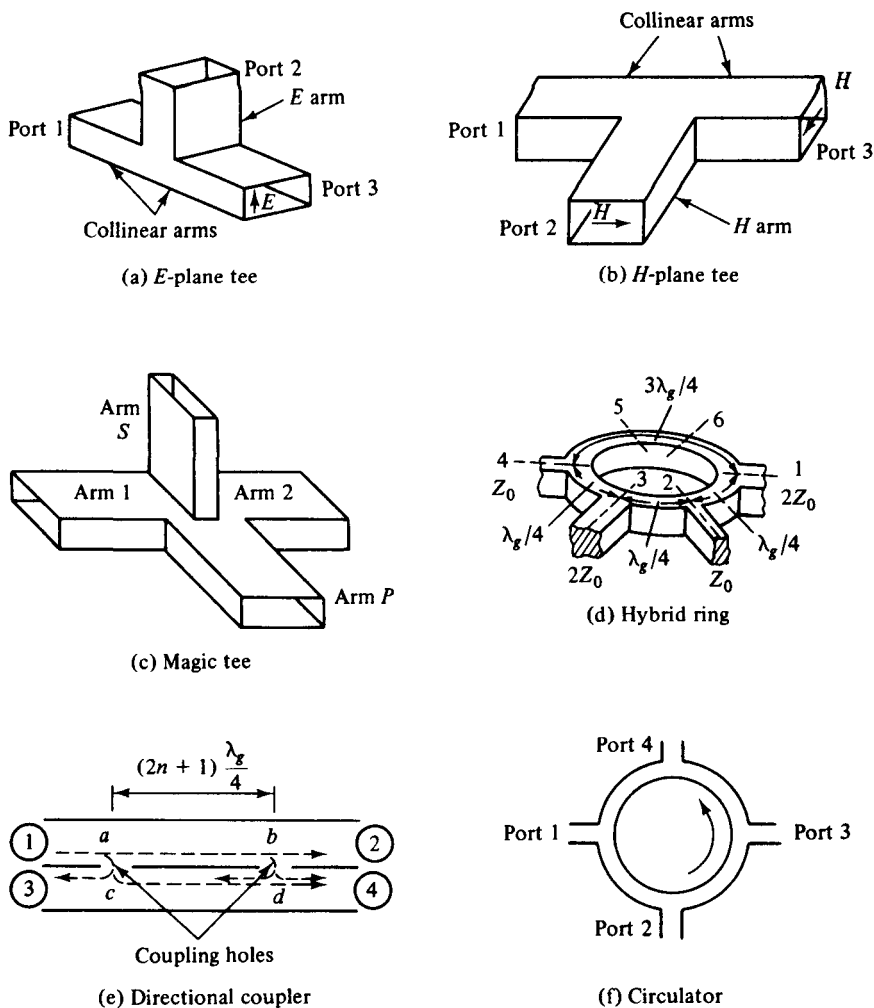
(f) Circulator

Figure 4-4-1   Microwave hybrids. (a) E-plane tee. (b) H-plane tee. (c) Magic tee. (d) Hybrid ring. (e) Directional coupler. (f) Circulator.
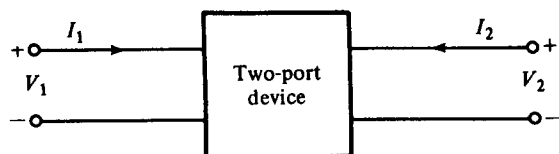
Figure 4-4-2 Two-port network.

Z parameters: $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

$$V_1 = z_{11}I_1 + z_{12}I_2 \qquad (4\text{-}4\text{-}5)$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \qquad (4\text{-}4\text{-}6)$$

$ABCD$ parameters: $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$

$$V_1 = AV_2 - BI_2 \qquad (4\text{-}4\text{-}7)$$
$$I_1 = CV_2 - DI_2 \qquad (4\text{-}4\text{-}8)$$

All these network parameters relate total voltages and total currents at each of the two ports. For instance,

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0} \qquad \text{(short circuit)} \qquad (4\text{-}4\text{-}9)$$

$$h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0} \qquad \text{(open circuit)} \qquad (4\text{-}4\text{-}10)$$

If the frequencies are in the microwave range, however, the H, Y, and Z parameters cannot be measured for the following reasons:

1. Equipment is not readily available to measure total voltage and total current at the ports of the network.
2. Short and open circuits are difficult to achieve over a broad band of frequencies.
3. Active devices, such as power transistors and tunnel diodes, frequently will not have stability for a short or open circuit.

Consequently, some new method of characterization is needed to overcome these problems. The logical variables to use at the microwave frequencies are traveling waves rather than total voltages and total currents. These are the S parameters, which are expressed as

$$b_1 = S_{11}a_1 + S_{12}a_2 \qquad (4\text{-}4\text{-}11a)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \qquad (4\text{-}4\text{-}11b)$$

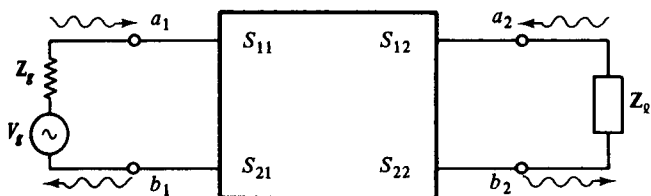Figure 4-4-3 shows the S parameters of a two-port network.

Figure 4-4-3 Two-port network.

4-4-1 Waveguide Tees

As noted, waveguide tees may consist of the E-plane tee, H-plane tee, magic tee, hybrid rings, corners, bends, and twists. All such waveguide components are discussed in this section.

Tee junctions.   In microwave circuits a waveguide or coaxial-line junction with three independent ports is commonly referred to as a *tee junction*. From the S-parameter theory of a microwave junction it is evident that a tee junction should be characterized by a matrix of third order containing nine elements, six of which should be independent. The characteristics of a three-port junction can be explained by three theorems of the tee junction. These theorems are derived from the equivalent-circuit representation of the tee junction. Their statements follow

1. A short circuit may always be placed in one of the arms of a three-port junction in such a way that no power can be transferred through the other two arms.
2. If the junction is symmetric about one of its arms, a short circuit can always be placed in that arm so that no reflections occur in power transmission between the other two arms. (That is, the arms present matched impedances.)
3. It is impossible for a general three-port junction of arbitrary symmetry to present matched impedances at all three arms.

The E-plane tee and H-plane tee are described below.

E-plane tee (series tee).   An E-plane tee is a waveguide tee in which the axis of its side arm is parallel to the E field of the main guide (see Fig. 4-4-4). If the collinear arms are symmetric about the side arm, there are two different transmission characteristics (see Fig. 4-4-5). It can be seen from Fig. 4-4-4 that if the E-plane tee is perfectly matched with the aid of screw tuners or inductive or capacitive windows at the junction, the diagonal components of the scattering matrix, S_{11}, S_{22}, and S_{33}, are zero because there will be no reflection. When the waves are fed into the side arm (port 3), the waves appearing at port 1 and port 2 of the collinear arm will be in opposite phase and in the same magnitude. Therefore

$$S_{13} = -S_{23} \qquad (4\text{-}4\text{-}12)$$

It should be noted that Eq. (4-4-12) does not mean that S_{13} is always positive and S_{23}

Port 3

Port 2

Side arm

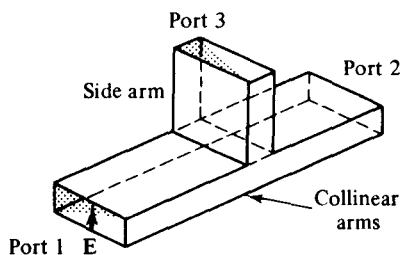Collinear arms

Port 1 E

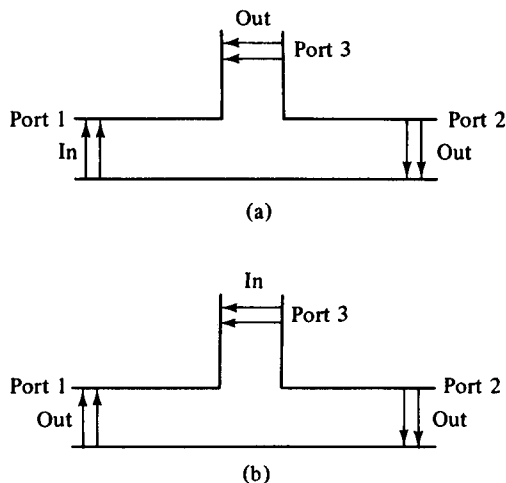**Figure 4-4-4** E-plane tee

Figure 4-4-5 Two-way transmission of E-plane tee. (a) Input through main arm. (b) Input from side arm.

is always negative. The negative sign merely means that S_{13} and S_{23} have opposite signs. For a matched junction, the **S** matrix is given by

$$\mathbf{S} = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix} \tag{4-4-13}$$

From the symmetry property of **S** matrix, the symmetric terms in Eq. (4-4-13) are equal and they are

$$S_{12} = S_{21} \qquad S_{13} = S_{31} \qquad S_{23} = S_{32} \tag{4-4-14}$$

From the zero property of **S** matrix, the sum of the products of each term of any column (or row) multiplied by the complex conjugate of the corresponding terms of any other column (or row) is zero and it is

$$S_{11} S_{12}^* + S_{21} S_{22}^* + S_{31} S_{32}^* = 0 \tag{4-4-15}$$

Hence

$$S_{13} S_{23}^* = 0 \tag{4-4-16}$$

This means that either S_{13} or S_{23}^*, or both, should be zero. However, from the unity property of **S** matrix, the sum of the products of each term of any one row (or column) multiplied by its complex conjugate is unity; that is,

$$S_{21} S_{21}^* + S_{31} S_{31}^* = 1 \tag{4-4-17}$$

$$S_{12} S_{12}^* + S_{32} S_{32}^* = 1 \tag{4-4-18}$$

$$S_{13} S_{13}^* + S_{23} S_{23}^* = 1 \tag{4-4-19}$$

Substitution of Eq. (4-4-14) in (4-4-17) results in

$$|S_{12}|^2 = 1 - |S_{13}|^2 = 1 - |S_{23}|^2 \tag{4-4-20}$$

Equations (4-4-19) and (4-4-20) are contradictory, for if $S_{13} = 0$, then S_{23} is also

zero and thus Eq. (4-4-19) is false. In a similar fashion, if $S_{23} = 0$, then S_{13} becomes zero and therefore Eq. (4-4-20) is not true. This inconsistency proves the statement that the tee junction cannot be matched to the three arms. In other words, the diagonal elements of the **S** matrix of a tee junction are not all zeros.

In general, when an E-plane tee is constructed of an empty waveguide, it is poorly matched at the tee junction. Hence $S_{ij} \neq 0$ if $i = j$. However, since the collinear arm is usually symmetric about the side arm, $|S_{13}| = |S_{23}|$ and $S_{11} = S_{22}$. Then the **S** matrix can be simplified to

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & -S_{13} \\ S_{13} & -S_{13} & S_{33} \end{bmatrix} \tag{4-4-21}$$

H-plane tee (shunt tee).   An H-plane tee is a waveguide tee in which the axis of its side arm is "shunting" the E field or parallel to the H field of the main guide as shown in Fig. 4-4-6.
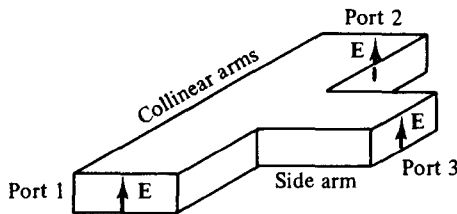
Figure 4-4-6 H-plane tee.

It can be seen that if two input waves are fed into port 1 and port 2 of the collinear arm, the output wave at port 3 will be in phase and additive. On the other hand, if the input is fed into port 3, the wave will split equally into port 1 and port 2 in phase and in the same magnitude. Therefore the **S** matrix of the H-plane tee is similar to Eqs. (4-4-13) and (4-4-21) except that

$$S_{13} = S_{23} \tag{4-4-22}$$

4-4-2 Magic Tees (Hybrid Tees)

A magic tee is a combination of the E-plane tee and H-plane tee (refer to Fig. 4-4-7). The magic tee has several characteristics:
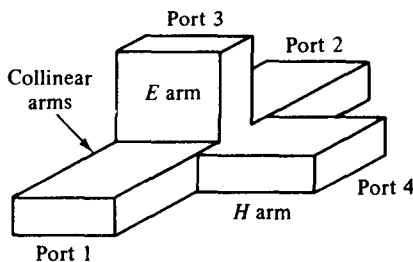

Figure 4-4-7   Magic tee.

1. If two waves of equal magnitude and the same phase are fed into port 1 and port 2, the output will be zero at port 3 and additive at port 4.
2. If a wave is fed into port 4 (the H arm), it will be divided equally between port 1 and port 2 of the collinear arms and will not appear at port 3 (the E arm).
3. If a wave is fed into port 3 (the E arm), it will produce an output of equal magnitude and opposite phase at port 1 and port 2. The output at port 4 is zero. That is, $S_{43} = S_{34} = 0$.
4. If a wave is fed into one of the collinear arms at port 1 or port 2, it will not appear in the other collinear arm at port 2 or port 1 because the E arm causes a phase delay while the H arm causes a phase advance. That is, $S_{12} = S_{21} = 0$.

Therefore the **S** matrix of a magic tee can be expressed as

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} \\ 0 & 0 & S_{23} & S_{24} \\ S_{31} & S_{32} & 0 & 0 \\ S_{41} & S_{42} & 0 & 0 \end{bmatrix} \tag{4-4-23}$$

The magic tee is commonly used for mixing, duplexing, and impedance measurements. Suppose, for example, there are two identical radar transmitters in equipment stock. A particular application requires twice more input power to an antenna than either transmitter can deliver. A magic tee may be used to couple the two transmitters to the antenna in such a way that the transmitters do not load each other. The two transmitters should be connected to ports 3 and 4, respectively, as shown in Fig. 4-4-8. Transmitter 1, connected to port 3, causes a wave to emanate from port 1 and another to emanate from port 2; these waves are equal in magnitude but opposite in phase. Similarly, transmitter 2, connected to port 4, gives rise to a wave at port 1 and another at port 2, both equal in magnitude and in phase. At port 1 the two opposite waves cancel each other. At port 2 the two in-phase waves add together; so double output power at port 2 is obtained for the antenna as shown in Fig. 4-4-8.
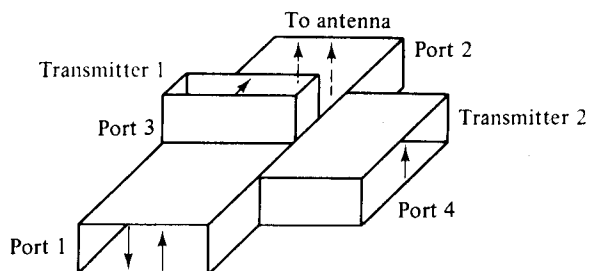

Figure 4-4-8 Magic tee-coupled transmitters to antenna.

4-4-3 Hybrid Rings (Rat-Race Circuits)

A hybrid ring consists of an annular line of proper electrical length to sustain standing waves, to which four arms are connected at proper intervals by means of series or parallel junctions. Figure 4-4-9 shows a hybrid ring with series junctions.
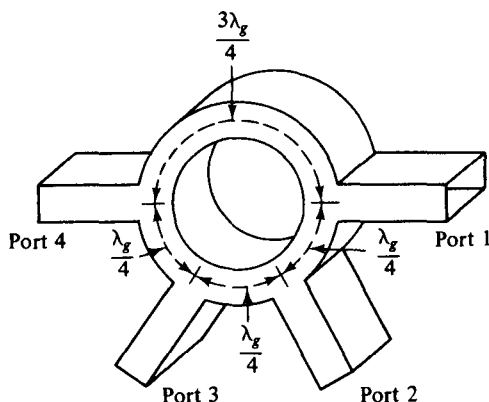
$$\frac{3\lambda_g}{4}$$

Port 4 $\frac{\lambda_g}{4}$ $\frac{\lambda_g}{4}$ Port 1

$$\frac{\lambda_g}{4}$$

Port 3   Port 2   **Figure 4-4-9**   Hybrid ring.

The hybrid ring has characteristics similar to those of the hybrid tee. When a wave is fed into port 1, it will not appear at port 3 because the difference of phase shifts for the waves traveling in the clockwise and counterclockwise directions is 180°. Thus the waves are canceled at port 3. For the same reason, the waves fed into port 2 will not emerge at port 4 and so on.

The **S** matrix for an ideal hybrid ring can be expressed as

$$\mathbf{S} = \begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{21} & 0 & S_{23} & 0 \\ 0 & S_{32} & 0 & S_{34} \\ S_{41} & 0 & S_{43} & 0 \end{bmatrix} \qquad (4\text{-}4\text{-}24)$$

It should be noted that the phase cancellation ocurs only at a designated frequency for an ideal hybrid ring. In actual hybrid rings there are small leakage couplings, and therefore the zero elements in the matrix of Eq. (4-4-24) are not quite equal to zero.

4-4-4 Waveguide Corners, Bends, and Twists

The waveguide corner, bend, and twist are shown in Fig. 4-4-10. These waveguide components are normally used to change the direction of the guide through an arbitrary angle.

In order to minimize reflections from the discontinuities, it is desirable to have the mean length L between continuities equal to an odd number of quarter-wavelengths. That is,

$$L = (2n + 1)\frac{\lambda_g}{4} \qquad (4\text{-}4\text{-}25)$$

where $n = 0, 1, 2, 3, \ldots$, and λ_g is the wavelength in the waveguide. If the mean length L is an odd number of quarter wavelengths, the reflected waves from both ends of the waveguide section are completely canceled. For the waveguide bend, the minimum radius of curvature for a small reflection is given by Southworth [2] as

$$R = 1.5b \qquad \text{for an } E \text{ bend} \qquad (4\text{-}4\text{-}26)$$

$$R = 1.5a \qquad \text{for an } H \text{ bend} \qquad (4\text{-}4\text{-}27)$$

where a and b are the dimensions of the waveguide bend as illustrated in Fig. 4-4-10.
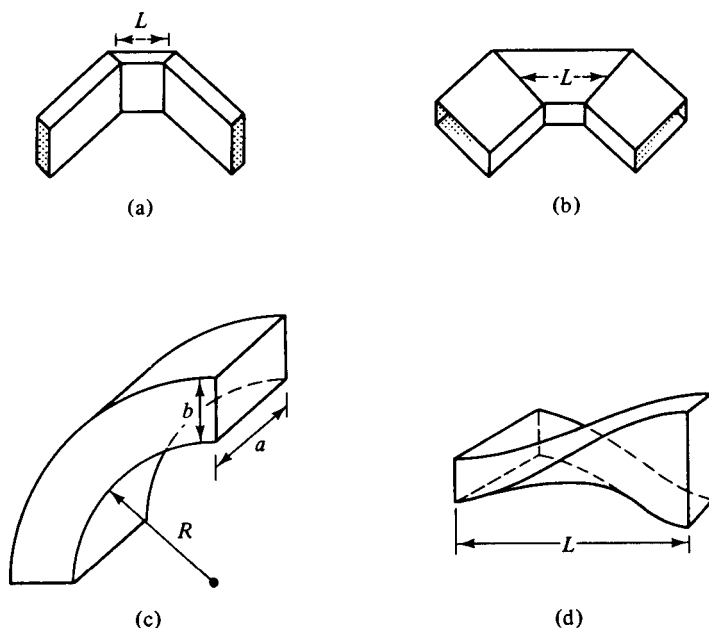

(a)

(b)

(c)

(d)

Figure 4-4-10   Waveguide corner, bend, and twist. (a) E-plane corner. (b) H-plane corner. (c) Bend. (d) Continuous twist.

4-5 DIRECTIONAL COUPLERS

A *directional coupler* is a four-port waveguide junction as shown in Fig. 4-5-1. It consists of a primary waveguide 1–2 and a secondary waveguide 3–4. When all ports are terminated in their characteristic impedances, there is free transmission of power, without reflection, between port 1 and port 2, and there is no transmission of power between port 1 and port 3 or between port 2 and port 4 because no coupling exists between these two pairs of ports. The degree of coupling between port 1 and port 4 and between port 2 and port 3 depends on the structure of the coupler.

The characteristics of a directional coupler can be expressed in terms of its coupling factor and its directivity. Assuming that the wave is propagating from port 1 to port 2 in the primary line, the coupling factor and the directivity are defined,
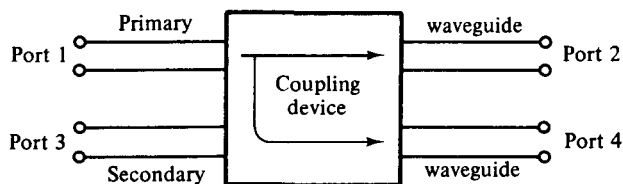

Figure 4-5-1   Directional coupler.

respectively, by

$$\text{Coupling factor (dB)} = 10 \log_{10} \frac{P_1}{P_4} \qquad (4\text{-}5\text{-}1)$$

$$\text{Directivity (dB)} = 10 \log_{10} \frac{P_4}{P_3} \qquad (4\text{-}5\text{-}2)$$

where P_1 = power input to port 1
$\quad\;\; P_3$ = power output from port 3
$\quad\;\; P_4$ = power output from port 4

It should be noted that port 2, port 3, and port 4 are terminated in their characteristic impedances. *The coupling factor is a measure of the ratio of power levels* in the primary and secondary lines. Hence if the coupling factor is known, a fraction of power measured at port 4 may be used to determine the power input at port 1. This significance is desirable for microwave power measurements because no disturbance, which may be caused by the power measurements, occurs in the primary line. The directivity is a measure of how well the forward traveling wave in the primary waveguide couples only to a specific port of the secondary waveguide. An ideal directional coupler should have infinite directivity. In other words, the power at port 3 must be zero because port 2 and port 4 are perfectly matched. Actually, well-designed directional couplers have a directivity of only 30 to 35 dB.

Several types of directional couplers exist, such as a two-hole directional couler, four-hole directional coupler, reverse-coupling directional coupler (Schwinger coupler), and Bethe-hole directional coupler (refer to Fig. 4-5-2). Only the very commonly used two-hole directional coupler is described here.
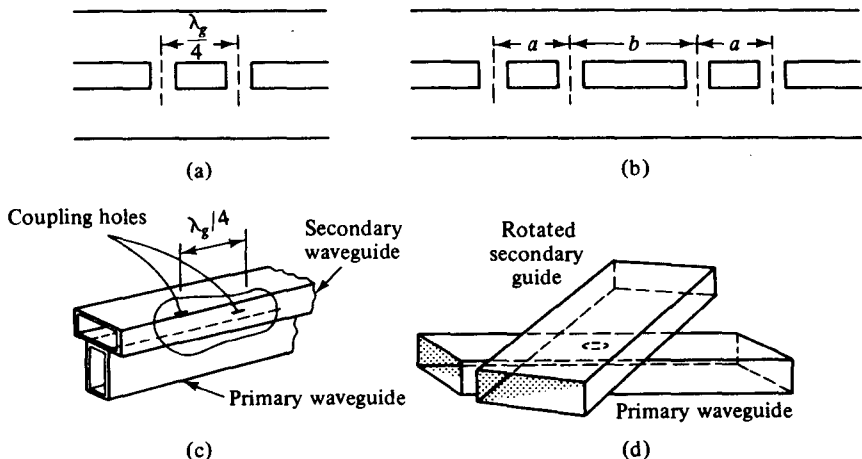

Figure 4-5-2 Different directional couplers. (a) Two-hole directional coupler. (b) Four-hole directional coupler. (c) Schwinger coupler. (d) Bethe-hole directional coupler.

4-5-1 Two-Hole Directional Couplers

A two-hole directional coupler with traveling waves propagating in it is illustrated in Fig. 4-5-3. The spacing between the centers of two holes must be

$$L = (2n + 1)\frac{\lambda_g}{4} \qquad (4\text{-}5\text{-}3)$$
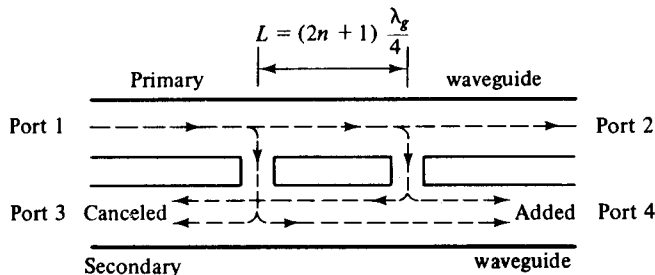
where n is any positive integer.

Figure 4-5-3 Two-hole directional coupler.

A fraction of the wave energy entered into port 1 passes through the holes and is radiated into the secondary guide as the holes act as slot antennas. The forward waves in the secondary guide are in the same phase, regardless of the hole space, and are added at port 4. The backward waves in the secondary guide (waves are progressing from right to left) are out of phase by $(2L/\lambda_g)2\pi$ rad and are canceled at port 3.

4-5-2 S Matrix of a Directional Coupler

In a directional coupler all four ports are completely matched. Thus the diagonal elements of the S matrix are zeros and

$$S_{11} = S_{22} = S_{33} = S_{44} = 0 \qquad (4\text{-}5\text{-}4)$$

As noted, there is no coupling between port 1 and port 3 and between port 2 and port 4. Thus

$$S_{13} = S_{31} = S_{24} = S_{42} = 0 \qquad (4\text{-}5\text{-}5)$$

Consequently, the S matrix of a directional coupler becomes

$$\mathbf{S} = \begin{bmatrix} 0 & S_{12} & 0 & S_{14} \\ S_{21} & 0 & S_{23} & 0 \\ 0 & S_{32} & 0 & S_{34} \\ S_{41} & 0 & S_{43} & 0 \end{bmatrix} \qquad (4\text{-}5\text{-}6)$$

Equation (4-5-6) can be further reduced by means of the zero property of the S matrix, so we have

$$S_{12}S_{14}^* + S_{32}S_{34}^* = 0 \qquad (4\text{-}5\text{-}7)$$

$$S_{21} S_{23}^* + S_{41} S_{43}^* = 0 \qquad (4\text{-}5\text{-}8)$$

Also from the unity property of the **S** matrix, we can write

$$S_{12} S_{12}^* + S_{14} S_{14}^* = 1 \qquad (4\text{-}5\text{-}9)$$

Equations (4-5-7) and (4-5-8) can also be written

$$|S_{12}||S_{14}| = |S_{32}||S_{34}| \qquad (4\text{-}5\text{-}10)$$

$$|S_{21}||S_{23}| = |S_{41}||S_{43}| \qquad (4\text{-}5\text{-}11)$$

Since $S_{12} = S_{21}$, $S_{14} = S_{41}$, $S_{23} = S_{32}$, and $S_{34} = S_{43}$, then

$$|S_{12}| = |S_{34}| \qquad (4\text{-}5\text{-}12)$$

$$|S_{14}| = |S_{23}| \qquad (4\text{-}5\text{-}13)$$

Let

$$S_{12} = S_{34} = p \qquad (4\text{-}5\text{-}14)$$

where p is positive and real. Then from Eq. (4-5-8)

$$p(S_{23}^* + S_{41}) = 0 \qquad (4\text{-}5\text{-}15)$$

Let

$$S_{23} = S_{41} = jq \qquad (4\text{-}5\text{-}16)$$

where q is positive and real. Then from Eq. (4-5-9)

$$p^2 + q^2 = 1 \qquad (4\text{-}5\text{-}17)$$

The **S** matrix of a directional coupler is reduced to

$$\mathbf{S} = \begin{bmatrix} 0 & p & 0 & jq \\ p & 0 & jq & 0 \\ 0 & jq & 0 & p \\ jq & 0 & p & 0 \end{bmatrix} \qquad (4\text{-}5\text{-}18)$$

Example 4-5-1:   Directional Coupler

A symmetric directional coupler with infinite directivity and a forward attenuation of 20 dB is used to monitor the power delivered to a load $\mathbf{Z}_\ell$ (see Fig. 4-5-4). Bolometer 1 introduces a VSWR of 2.0 on arm 4; bolometer 2 is matched to arm 3. If bolometer 1
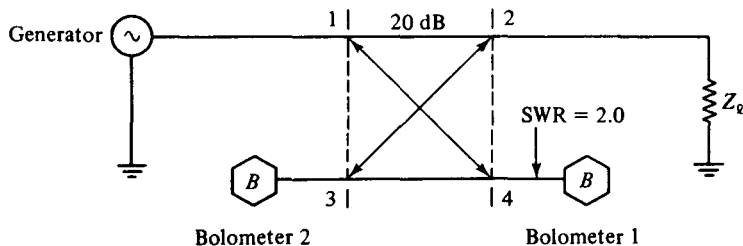

Figure 4-5-4 Power measurements by directional coupler.

reads 8 mW and bolometer 2 reads 2 mW, find: **(a)** the amount of power dissipated in the load Z_ℓ; **(b)** the VSWR on arm 2.

Solution   The wave propagation in the directional coupler is shown in Fig. 4-5-5.
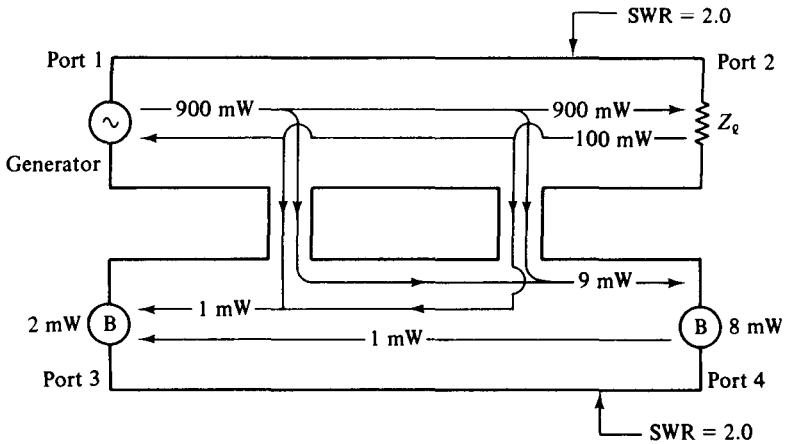

Figure 4-5-5   Wave propagation in the directional coupler.

a. Power dissipation at Z_ℓ.

1. The reflection coefficient at port 4 is

$$|\Gamma| = \frac{\rho - 1}{\rho + 1} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$$

2. Since the incident power and reflected power are related by

$$P^- = P^+ |\Gamma|^2$$

where P^+ = incident power and P^- = reflected power, then

$$|\Gamma| = \frac{1}{3} = \sqrt{\frac{P^-}{P^+}} = \sqrt{\frac{P^-}{8 + P^-}}$$

The incident power to port 4 is $P_4^+ = 9$ mW, and the reflected power from port 4 is $P_4^- = 1$ mW.

3. Since port 3 is matched and the bolometer at port 3 reads 2 mW, then 1 mW must be radiated through the holes.

4. Since 20 dB is equivalent to a power ratio of 100:1, the power intput at port 1 is given by

$$P_1 = 100 P_4^+ = 900 \text{ mW}$$

and the power reflected from the load is

$$P_2^- = 100 \times (1 \text{ mW}) = 100 \text{ mW}$$

5. The power dissipated in the load is

$$P_\ell = P_2^+ - P_2^- = 900 - 100 = 800 \text{ mW}$$

b. The reflection coefficient is calculated as

$$|\Gamma| = \sqrt{\frac{P^-}{P^+}} = \sqrt{\frac{100}{900}} = \frac{1}{3}$$

Then the VSWR on arm 2 is

$$\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2.0$$

4-5-3 Hybrid Couplers

Hybrid couplers are interdigitated microstrip couplers consisting of four parallel strip lines with alternate lines tied together. A single ground plane, a single dielectric, and a single layer of metallization are used. This type of coupler, called a Lange hybrid coupler [3], has four ports, as shown in Fig. 4-5-6.
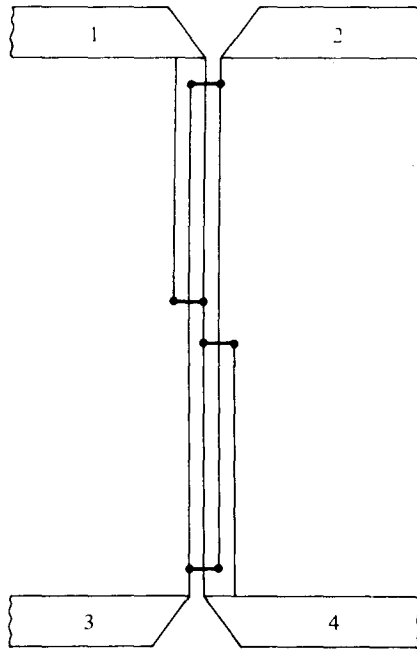

Figure 4-5-6 Lange hybrid coupler.

A signal wave incident in port 1 couples equal power into ports 2 and 4, but none into port 3. There are two basic types of Lange couplers: 180° hybrids and 90° (quadrature) hybrids. The latter are also called 3-dB directional couplers.

Hybrid couplers are frequently used as components in microwave systems or subsystems such as attenuators, balanced amplifiers, balanced mixers, modulators, discriminators, and phase shifters. The hybrid has a directivity of over 27 dB, a return loss of over 25 dB, an insertion loss of less than 0.13 dB, and an imbalance of less than 0.25 dB over a 40% bandwidth.

In modern microwave circuit design, Lange hybrid couplers are commonly

used in balanced amplifier circuitry for high-power and broad-bandwidth applications, as shown in Fig. 4-5-7.

Single-stage or cascaded double-stage GaAs MESFET chips are connected in parallel to two 3-dB and 90-degree Lange hybrid couplers. Their basic relationship can be expressed by the following three equations:

$$S_{11} = \tfrac{1}{2}(S_{11a} - S_{11b}) \tag{4-5-19}$$

$$S_{22} = \tfrac{1}{2}(S_{22a} - S_{22b}) \tag{4-5-20}$$

and

$$\text{Gain} = |S_{21}|^2 = \tfrac{1}{4}|S_{21a} + S_{21b}|^2 \tag{4-5-21}$$

where a and b indicate the two GaAs MESFET chips, and 1 and 2 refer to the input and output ports, respectively. The VSWRs of the balanced amplifier can be expressed as

$$\text{VSWR} = \frac{1 + |S_{11}|}{1 - |S_{11}|} \qquad \text{for the input port} \tag{4-5-22}$$

and

$$\text{VSWR} = \frac{1 + |S_{22}|}{1 - |S_{22}|} \qquad \text{for the output port} \tag{4-5-23}$$
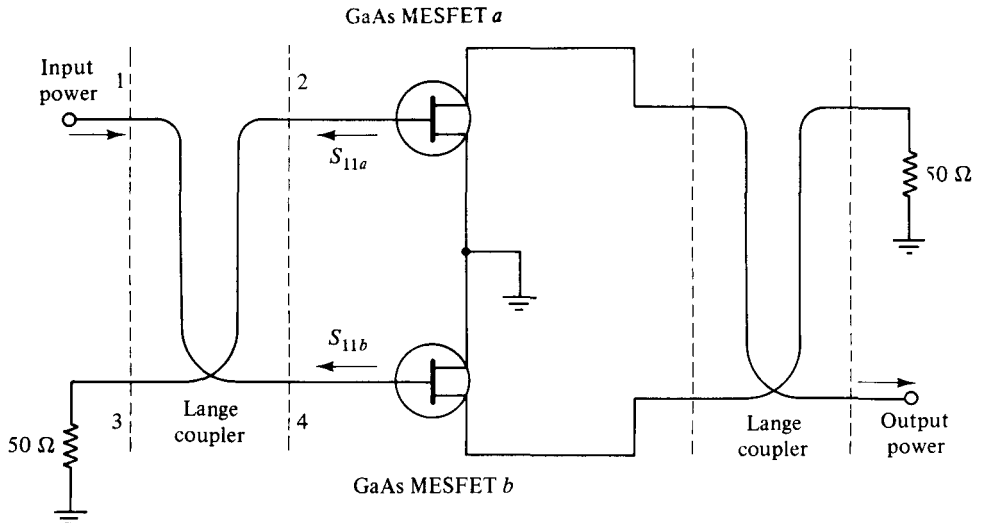

Figure 4-5-7   Balanced amplifier with Lange couplers.

Theoretically, if the two GaAs MESFET chips (or four chips in a double-stage amplifier circuit) are identical, the amplifier is balanced and its VSWR will be unity. Practically, however, characteristics of the two GaAs MESFET chips are not actually measured and they may not be the same. When their characteristics are different, the amplifier will not be balanced and manual tuning will be needed to balance it. Therefore, for mass production it is necessary to characterize the GaAs MESFET

chips in advance before placing them in the microwave integrated circuit in order to minimize the tuning work, reduce the production cost, and increase the hybrid reproducibility.

Example 4-5-2: Operation of a Balanced Amplifier

A GaAs MESFET balanced amplifier with two Lange couplers has the following parameters:

S parameters:	$S_{11a} = S_{11b}$
	$S_{22a} = S_{22b}$
Input signal power:	$P_{in} = 200$ mW
Power gain of each GaAs chip:	Gain $= 10$ dB

Determine: **(a)** the input and output VSWRs; **(b)** the output power in watts; **(c)** the linear output power gain in dB.

Solution

a. From Eqs. (4-5-22) and (4-5-23), the input and output VSWRs are unity.
b. The output power is

$$P_{out} = 200 \times 10 \times 2 = 4000 \text{ mW} = 4 \text{ W}$$

c. Because two GaAs chips are in parallel, the linear output power gain is

$$\text{Gain} = 10 \log (2) = 3 \text{ dB}$$

4-6 CIRCULATORS AND ISOLATORS

Both microwave circulators and microwave isolators are nonreciprocal transmission devices that use the property of Faraday rotation in the ferrite material. In order to understand the operating principles of circulators and isolators, let us describe the behavior of ferrites in the nonreciprocal phase shifter.

A *nonreciprocal phase shifter* consists of a thin slab of ferrite placed in a rectangular waveguide at a point where the dc magnetic field of the incident wave mode is circularly polarized. Ferrite is a family of $MeO \cdot Fe_2O_3$, where Me is a divalent iron metal. When a piece of ferrite is affected by a dc magnetic field, the ferrite exhibits Faraday rotation. It does so because the ferrite is nonlinear material and its permeability is an asymmetric tensor [4], as expressed by

$$\mathbf{B} = \hat{\mu}\mathbf{H} \tag{4-6-1}$$

where

$$\hat{\mu} = \mu_0(1 + \hat{\chi}_m) \tag{4-6-2}$$

$$\hat{\chi}_m = \begin{bmatrix} \chi_m & j\kappa & 0 \\ j\kappa & \chi_m & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4-6-3}$$

which is the tensor magnetic susceptibility. Here χ is the diagonal susceptibility and κ is the off-diagonal susceptibility.

When a dc magnetic field is applied to a ferrite, the unpaired electrons in the ferrite material tend to line up with the dc field because of their magnetic dipole moment. However, the nonreciprocal precession of unpaired electrons in the ferrite causes their relative permeabilities (μ_r^+, μ_r^-) to be unequal and the wave in the ferrite is then circularly polarized. The propagation constant for a linearly polarized wave inside the ferrite can be expressed as [4]

$$\gamma^\pm = j\omega\sqrt{\epsilon\mu_0(\mu + \kappa)} \qquad (4\text{-}6\text{-}4)$$

where

$$\mu = 1 + \hat{\chi}_m \qquad (4\text{-}6\text{-}5)$$

$$\mu_r^+ = \mu + \kappa \qquad (4\text{-}6\text{-}6)$$

$$\mu_r^- = \mu - \kappa \qquad (4\text{-}6\text{-}7)$$

The relative permeability μ_r changes with the applied dc magnetic field as given by

$$\mu_r^\pm = 1 + \frac{\gamma_e M_e}{|\gamma_e| H_{dc} \mp \omega} \qquad (4\text{-}6\text{-}8)$$

where γ_e = gyromagnetic ratio of an electron

M_e = saturation magnetization

ω = angular frequency of a microwave field

H_{dc} = dc magnetic field

μ_r^+ = relative permeability in the clockwise direction (right or positive circular polarization)

μ_r^- = relative permeability in the counterclockwise direction (left or negative circular polarization)

It can be seen from Eq. (4-6-8) that if $\omega = |\gamma_e| H_{dc}$, then μ_r^+ is infinite. This phenomenon is called the *gyromagnetic resonance* of the ferrite. A graph of μ_r is plotted as a function of H_{dc} for longitudinal propagation in Fig. 4-6-1.

If μ_r^+ is much larger than μ_r^- ($\mu_r^+ \gg \mu_r^-$), the wave in the ferrite is rotated in
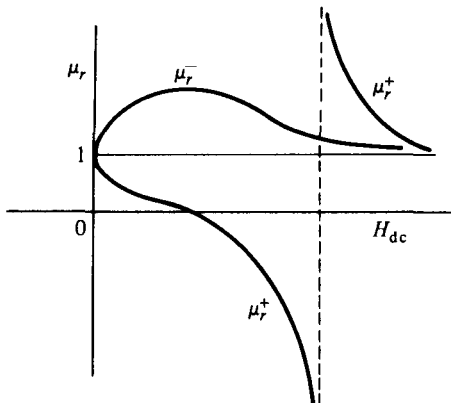

Figure 4-6-1   Curves of μ_r versus H_{dc} for axial propagation.

the clockwise direction. Consequently, the propagation phase constant β^+ for the forward direction differs from the propagation phase constant β^- for the backward direction. By choosing the length of the ferrite slab and the dc magnetic field so that

$$\omega = (\beta^+ - \beta^-)\ell = \frac{\pi}{2} \qquad (4\text{-}6\text{-}9)$$

a differential phase shift of 90° for the two directions of propagation can be obtained.

4-6-1 Microwave Circulators

A *microwave circulator* is a multiport waveguide junction in which the wave can flow only from the nth port to the $(n + 1)$th port in one direction (see Fig. 4-6-2). Although there is no restriction on the number of ports, the four-port microwave circulator is the most common. One type of four-port microwave circulator is a combination of two 3-dB side-hole directional couplers and a rectangular waveguide with two nonreciprocal phase shifters as shown in Fig. 4-6-3.
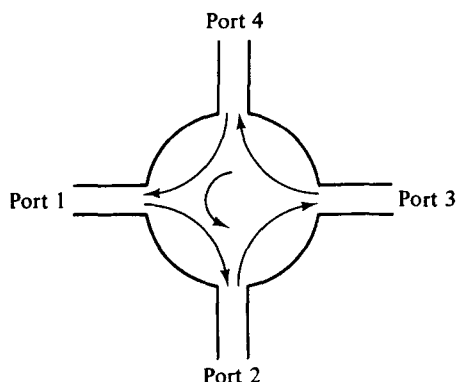

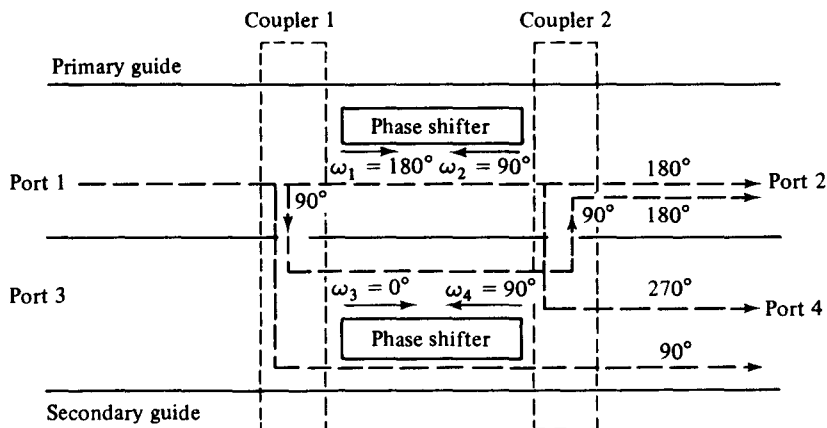**Figure 4-6-2** The symbol of a circulator.

Figure 4-6-3 Schematic diagram of four-port circulator.

The operating principle of a typical microwave circulator can be analyzed with the aid of Fig. 4-6-3. Each of the two 3-dB couplers in the circulator introduces a phase shift of 90°, and each of the two phase shifters produces a certain amount of phase change in a certain direction as indicated. When a wave is incident to port 1, the wave is split into two components by coupler 1. The wave in the primary guide arrives at port 2 with a relative phase change of 180°. The second wave propagates through the two couplers and the secondary guide and arrives at port 2 with a relative phase shift of 180°. Since the two waves reaching port 2 are in phase, the power transmission is obtained from port 1 to port 2. However, the wave propagates through the primary guide, phase shifter, and coupler 2 and arrives at port 4 with a phase change of 270°. The wave travels through coupler 1 and the secondary guide, and it arrives at port 4 with a phase shift of 90°. Since the two waves reaching port 4 are out of phase by 180°, the power transmission from port 1 to port 4 is zero. In general, the differential propagation constants in the two directions of propagation in a waveguide containing ferrite phase shifters should be

$$\omega_1 - \omega_3 = (2m + 1)\pi \qquad \text{rad/s} \qquad (4\text{-}6\text{-}10)$$

$$\omega_2 - \omega_4 = 2n\pi \qquad \text{rad/s} \qquad (4\text{-}6\text{-}11)$$

where m and n are any integers, including zeros. A similar analysis shows that a wave incident to port 2 emerges at port 3 and so on. As a result, the sequence of power flow is designated as $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$.

Many types of microwave circulators are in use today. However, their principles of operation remain the same. Figure 4-6-4 shows a four-port circulator constructed of two magic tees and a phase shifter. The phase shifter produces a phase shift of 180°. The explanation of how this circulator works is left as an exercise for the reader.
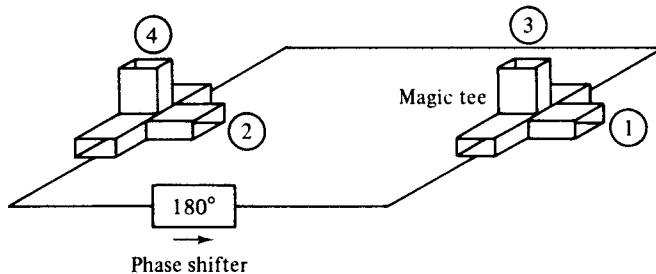

Figure 4-6-4   A four-port circulator.

A perfectly matched, lossless, and nonreciprocal four-port circulator has an **S** matrix of the form

$$\mathbf{S} = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} \\ S_{21} & 0 & S_{23} & S_{24} \\ S_{31} & S_{32} & 0 & S_{34} \\ S_{41} & S_{42} & S_{43} & 0 \end{bmatrix} \qquad (4\text{-}6\text{-}12)$$

Using the properties of S parameters as described previously, the **S** matrix in Eq.

(4-6-12) can be simplified to

$$
\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\tag{4-6-13}
$$

4-6-2 Microwave Isolators

An *isolator* is a nonreciprocal transmission device that is used to isolate one component from reflections of other components in the transmission line. An ideal isolator completely absorbs the power for propagation in one direction and provides lossless transmission in the opposite direction. Thus the isolator is usually called *uniline*. Isolators are generally used to improve the frequency stability of microwave generators, such as klystrons and magnetrons, in which the reflection from the load affects the generating frequency. In such cases, the isolator placed between the generator and load prevents the reflected power from the unmatched load from returning to the generator. As a result, the isolator maintains the frequency stability of the generator.

Isolators can be constructed in many ways. They can be made by terminating ports 3 and 4 of a four-port circulator with matched loads. On the other hand, isolators can be made by inserting a ferrite rod along the axis of a rectangular waveguide as shown in Fig. 4-6-5. The isolator here is a Faraday-rotation isolator. Its operating principle can be explained as follows [5]. The input resistive card is in the y-z plane, and the output resistive card is displaced 45° with respect to the input card. The dc magnetic field, which is applied longitudinally to the ferrite rod, rotates the wave plane of polarization by 45°. The degrees of rotation depend on the length and diameter of the rod and on the applied dc magnetic field. An input TE_{10} dominant mode is incident to the left end of the isolator. Since the TE_{10} mode wave is perpendicular to the input resistive card, the wave passes through the ferrite rod without attenuation. The wave in the ferrite rod section is rotated clockwise by 45° and is normal to the output resistive card. As a result of rotation, the wave arrives at the output
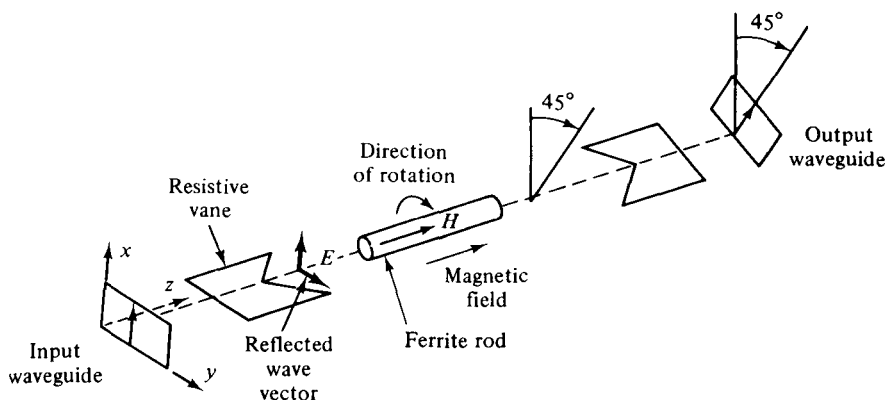

Figure 4-6-5   Faraday-rotation isolator.

end without attenuation at all. On the contrary, a reflected wave from the output end is similarly rotated clockwise 45° by the ferrite rod. However, since the reflected wave is parallel to the input resistive card, the wave is thereby absorbed by the input card. The typical performance of these isolators is about 1-dB insertion loss in forward transmission and about 20- to 30-dB isolation in reverse attenuation.

REFERENCES

[1] SAAD, T., and R. C. HANSEN, *Microwave Engineer's Handbook*, Vol. 1. Artech House, Dedham, Mass., 1971.

[2] SOUTHWORTH, G. E., *Principles and Applications of Waveguide Transmission*. Chapters 8 and 9. D. Van Nostrand Company, Princeton, N.J., 1950.

[3] LANGE, J. Interdigited stripline quadrature hybrid. *IEEE Trans. on Microwave Theory and Techniques,* **MTT-17,** No. 12, 1150–1151, December 1969.

[4] SOOHOO, R. F., *Theory and Applications of Ferrites*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1960.

[5] BOWNESS, C., Microwave ferrites and their applications. *Microwave J.,* **1,** 13–21, July-August 1958.

SUGGESTED READINGS

LIAO, S. Y., *Engineering Applications of Electromagnetic Theory*. West Publishing Company, St. Paul, Minn., 1988.

RIZZI, P. A., *Microwave Engineering: Passive Circuits*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1988.

PROBLEMS

Rectangular waveguides

4-1. An air-filled rectangular waveguide has dimensions of $a = 6$ cm and $b = 4$ cm. The signal frequency is 3 GHz. Compute the following for the TE_{10}, TE_{01}, TE_{11}, and TM_{11} modes:
   a. Cutoff frequency
   b. Wavelength in the waveguide
   c. Phase constant and phase velocity in the waveguide
   d. Group velocity and wave impedance in the waveguide

4-2. Show that the TM_{01} and TM_{10} modes in a rectangular waveguide do not exist.

4-3. The dominant mode TE_{10} is propagated in a rectangular waveguide of dimensions $a = 6$ cm and $b = 4$ cm. The distance between a maximum and a minimum is 4.47 cm. Determine the signal frequency of the dominant mode.

4-4. A TE_{11} mode of 10 GHz is propagated in an air-filled rectangular waveguide. The mag-

netic field in the z direction is given by

$$\mathbf{H}_z = \mathbf{H}_0 \cos \left(\frac{\pi x}{\sqrt{6}} \right) \cos \left(\frac{\pi y}{\sqrt{6}} \right) \qquad \text{A/m}$$

The phase constant is $\beta = 1.0475$ rad/cm, the quantities x and y are expressed in centimeters, and $a = b = \sqrt{6}$ are also in centimeters. Determine the cutoff frequency f_c, phase velocity v_g, guided wavelength λ_g, and the magnetic field intensity in the y direction.

4-5. A rectangular waveguide is designed to propagate the dominant mode TE_{10} at a frequency of 5 GHz. The cutoff frequency is 0.8 of the signal frequency. The ratio of the guide height to width is 2. The time-average power flowing through the guide is 1 kW. Determine the magnitudes of electric and magnetic intensities in the guide and indicate where these occur in the guide.

4-6. An air-filled rectangular waveguide has dimensions of $a = 6$ cm and $b = 4$ cm. The guide transports energy in the dominant mode TE_{10} at a rate of 1 horsepower (746 J). If the frequency is 20 GHz, what is the peak value of electric field occurring in the guide?

4-7. An impedance of $(0.5 - j0.4)Z_0$ is connected to a rectangular waveguide. A capacitive window with a susceptance $jB = j0.4Y_0$ is located at a distance of 0.2λ from the load.
  a. Determine the VSWR on the line in the absence of the window.
  b. Find the VSWR on the line in the presence of the window.

4-8. An air-filled rectangular waveguide with dimensions of 3 cm × 1 cm operates in the TE_{10} mode at 10 GHz. The waveguide is perfectly matched and the maximum E field existing everywhere in the guide is 10^3 V/m. Determine the voltage, current, and wave impedance in the waveguide.

4-9. The dominant mode TE_{10} is propagated in a rectangular waveguide of dimensions $a = 2.25$ cm and $b = 1$ cm. Assume an air dielectric with a breakdown gradient of 30 kV/cm and a frequency of 10 GHz. There are no standing waves in the guide. Determine the maximum average power that can be carried by the guide.

4-10. A rectangular waveguide is terminated in an unknown impedance at $z = 25$ cm. A dominant mode TE_{10} is propagated in the guide, and its VSWR is measured as 2.8 at a frequency of 8 GHz. The adjacent voltage minima are located at $z = 9.46$ cm and $z = 12.73$ cm.
  a. Determine the value of the load impedance in terms of Z_0.
  b. Find the position closest to the load where an inductive window is placed in order to obtain a VSWR of unity.
  c. Determine the value of the window admittance.

4-11. A rectangular waveguide is filled by dielectric material of $\epsilon_r = 9$ and has inside dimensions of 7 × 3.5 cm. It operates in the dominant TE_{10} mode.
  a. Determine the cutoff frequency.
  b. Find the phase velocity in the guide at a frequency of 2 GHz.
  c. Find the guided wavelength λ_g at the same frequency.

4-12. The electric field intensity of the dominant TE_{10} mode in a lossless rectangular waveguide is

$$E_y = E_0 \sin \left(\frac{\pi x}{a} \right) e^{-j\beta_g z} \qquad \text{for } f > f_c$$

  a. Find the magnetic field intensity $\mathbf{H}$.
  b. Compute the cutoff frequency and the time-average transmitted power.

Circular waveguides

4-13. An air-filled circular waveguide is to be operated at a frequency of 6 GHz and is to have dimensions such that $f_c = 0.8f$ for the dominant mode. Determine:
  a. The diameter of the guide
  b. The wavelength λ_g and the phase velocity v_g in the guide

4-14. An air-filled circular waveguide of 2 cm inside radius is operated in the TE_{01} mode.
  a. Compute the cutoff frequency.
  b. If the guide is to be filled with a dielectric material of $\epsilon_r = 2.25$, to what value must its radius be changed in order to maintain the cutoff frequency at its original value?

4-15. An air-filled circular waveguide has a radius of 1.5 cm and is to carry energy at a frequency of 10 GHz. Find all TE and TM modes for which transmission is possible.

4-16. A TE_{11} wave is propagating through a circular waveguide. The diameter of the guide is 10 cm, and the guide is air-filled.
  a. Find the cutoff frequency.
  b. Find the wavelength λ_g in the guide for a frequency of 3 GHz.
  c. Determine the wave impedance in the guide.

4-17. An air-filled circular waveguide has a diameter of 4 cm and is to carry energy at a frequency of 10 GHz. Determine all TE_{np} modes for which transmission is possible.

4-18. A circular waveguide has a cutoff frequency of 9 GHz in dominant mode.
  a. Find the inside diameter of the guide if it is air-filled.
  b. Determine the inside diameter of the guide if the guide is dielectric-filled. The relative dielectric constant is $\epsilon_r = 4$.

Microwave cavities

4-19. A coaxial resonator is constructed of a section of coaxial line and is open-circuited at both ends. The resonator is 5 cm long and filled with dielectric of $\epsilon_r = 9$. The inner conductor has a radius of 1 cm and the outer conductor has a radius of 2.5 cm.
  a. Find the resonant frequency of the resonator.
  b. Determine the resonant frequency of the same resonator with one end open and one end shorted.

4-20. An air-filled circular waveguide has a radius of 3 cm and is used as a resonator for TE_{01} mode at 10 GHz by placing two perfectly conducting plates at its two ends. Determine the minimum distance between the two end plates.

4-21. A four-port circulator is constructed of two magic tees and one phase shifter as shown in Fig. 4-6-4. The phase shifter produces a phase shift of 180°. Explain how this circulator works.

4-22. A coaxial resonator is constructed of a section of coaxial line 6 cm long and is short-circuited at both ends. The circular cavity has an inner radius of 1.5 cm and an outer radius of 3.5 cm. The line is dielectric-filled with $\epsilon_r = 2.25$.
  a. Determine the resonant frequency of the cavity for TEM_{001}.
  b. Calculate the quality Q of the cavity.

4-23. A rectangular-cavity resonator has dimensions of $a = 5$ cm, $b = 2$ cm, and $d = 15$ cm. Compute:
  a. The resonant frequency of the dominant mode for an air-filled cavity
  b. The resonant frequency of the dominant mode for a dielectric-filled cavity of $\epsilon_r = 2.56$

4-24. An undercoupled resonant cavity is connected to a lossless transmission line as shown in Fig. P4-24. The directional coupler is assumed to be ideal and matched on all arms. The unloaded Q of the cavity is 1000 and the VSWR at resonance is 2.5.
  a. Calculate the loaded Q_ℓ of the cavity.
  b. Find the reading of bolometer 2 if bolometer 1 reads 4 mW.
  c. Compute the power dissipated in the cavity.


P4-24
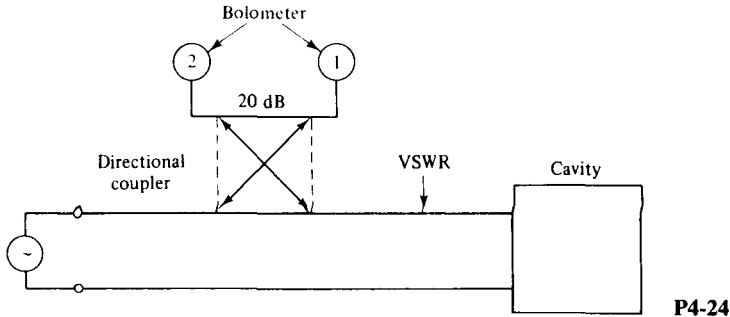
Hybrid circuits

4-25. A microwave transmission system consists of a generator, an overcoupled cavity, two ideal but not identical dual directional couplers with matched bolometers, and a load Z_ℓ. The lossless transmission line has a characteristic impedance Z_0. The readings of the four bolometers (1, 2, 3, and 4) are 2 mW, 4 mW, 0 and 1 mW, respectively. The system is shown in Fig. P4-25.
  a. Find the load impedance Z_ℓ in terms of Z_0.
  b. Calculate the power dissipated by Z_ℓ.
  c. Compute the power dissipated in the cavity.
  d. Determine the VSWR on the input transmission line.
  e. Find the ratio of Q_ℓ/Q_0 for the cavity.

P4-25
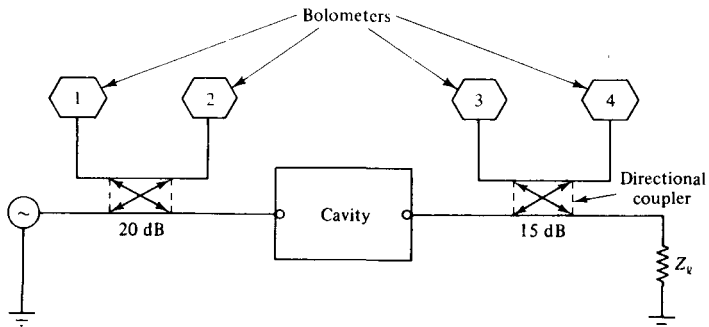
4-26. A symmetric directional coupler has an infinite directivity and a forward attenuation of 20 dB. The coupler is used to monitor the power delivered to a load Z_ℓ as shown in Fig. P4-26. Bolometer 1 introduces a VSWR of 2.0 on arm 1; bolometer 2 is matched to arm 2. If bolometer 1 reads 9 mW and bolometer 2 reads 3 mW:
  a. Find the amount of power dissipated in the load Z_ℓ.
  b. Determine the VSWR on arm 3.

Bolometer 2   VSWR = 2.0   Bolometer 1

4 20 dB 3

Z_k

P4-26
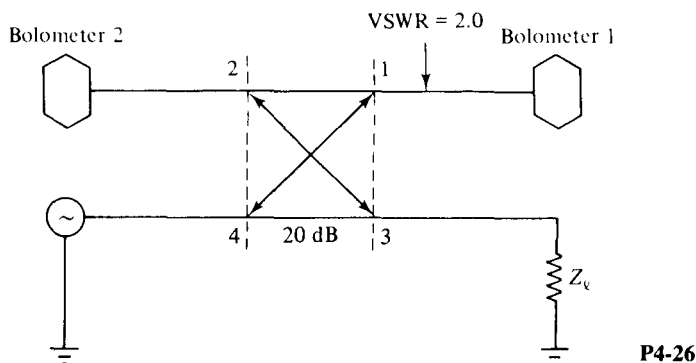
4-27. A semicircular-cavity resonator has a length of 5 cm and a radius of 2.5 cm.
  a. Calculate the resonant frequency for the dominant mode if the cavity is air-filled.
  b. Repeat part (a) if the cavity is loaded by a dielectric with a relative constant of 9.

4-28. The impedance matrix of a certain lumped-element network is given by

$$[z_{ij}] = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Determine the scattering matrix by using S-parameter theory and indicate the values of the components:

$$[S_{ki}] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

4-29. A hybrid waveguide is constructed of two identical rectangular waveguides across each other at the center and works as a four-port device. Write a general scattering matrix and then simplify it as much as possible by inspection of geometric symmetry and by use of the known phases of the electric waves.

4-30. A helical slow-wave structure has a pitch P of 2 mm and a diameter of 4 cm. Calculate the wave velocity in the axial direction of the helix.

4-31. Two 3-dB quadrature Lange couplers are used in a GaAs MESFET balanced amplifier circuit with the following parameters:

MESFET a: Reflection coefficients $S_{11a} = 0.7488\underline{/-158.3°}$

$S_{22a} = 0.8521\underline{/-155.7°}$

Forward transmission $S_{21a} = 1.3500\underline{/-8.5°}$
coefficient

MESFET b: Reflection coefficients $S_{11b} = 0.6210\underline{/-175.9°}$

$S_{22b} = 0.7727\underline{/-151.4°}$

Forward transmission $S_{21b} = 1.2200\underline{/-19.1°}$
coefficient

Compute: **a.** The input and output VSWRs
  b. The power gain in dB for the balanced amplifier
  c. The power loss in dB if one MESFET fails
  d. The linear output power gain in dB